A Unified Production and Matching Function: Implications for Factor Shares

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Abstract

This paper develops microfoundations for a unified aggregate production function. Labor market frictions are naturally built into the aggregate production function because matching and production are *two aspects of a single process.* Entrepreneurs with heterogeneous productivity levels hire capital and compete for workers. If no entrepreneurs approach a given worker he is unemployed, otherwise the entrepreneur with the highest productivity hires the worker. The model provides new insights into the behavior of factor shares. If the entrepreneurs' productivity distribution is Pareto, the aggregate production function is Cobb-Douglas *only in the limit* as frictional unemployment disappears. In this limiting case, factors are paid their marginal product and factor shares are constant. Outside this limit, factor shares are not generally constant, enabling us to examine their behavior. A key prediction of the model is that labor's share is *counter-cyclical* provided that workers' outside option is sufficiently high.

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1 Introduction

This paper develops microfoundations for a *unified* aggregate production function. It is "unified" in the sense that it incorporates an endogenous matching function which determines the unemployment rate. Labor market frictions are naturally built into this aggregate production function because matching and production are *two aspects of a single process*. The primary inputs determining aggregate output through this unified production and matching technology are (i) the capital rented by entrepreneurs, who may be either successful or unsuccessful in hiring workers, and (ii) the number of potential workers, who may end up either employed or unemployed.

Competition between entrepreneurs to hire workers drives both employment outcomes and aggregate labor productivity. Entrepreneurs with heterogeneous productivity levels compete for workers' labor. Successful entrepreneurs are those with the highest productivity approaching a given worker, and employed workers are those with at least one entrepreneur approaching them. This process generates an endogenous productivity distribution across workers that simultaneously incorporates two dimensions. First, there is the *productivity* effect of competition between entrepreneurs, which increases aggregate productivity by allocating labor towards more productive entrepreneurs. Second, there is the *employment* effect of the frictional matching process, which leads to unemployment for some workers.

I apply this unified framework to a question that has puzzled economists for decades: *Why* are factor shares relatively stable? More precisely, why is there no long-run trend in the relative shares of income accruing to capital and labor? This paper provides a partial answer to this question by considering *how* constant factor shares could possibly arise. At the same time, the model can explain the systematic variation in factor shares, in terms of both short-run cyclical fluctuations and medium-run shifts due to changes in labor market conditions.

There are two existing alternatives for accounting for the stability of factor shares.[1] The first is to assume that the aggregate production function is Cobb-Douglas. If factors are paid their marginal product, income shares for capital and labor are constant.

[1] Acemoglu (2003) states that "Almost all models of growth and capital accumulation, of both endogenous and exogenous types, explain the stability of factor shares using one of two assumptions: either the elasticity of substitution between capital and labor is taken to be equal to 1, or all technical change is assumed to be labor-augmenting (Harrod neutral)."

The second is to use a constant-elasticity-of-subsitution (CES) aggregate production function and assume that technological change is purely labor-augmenting. Along a balanced growth path in which capital deepening and technological change proceed at exactly the same rate, factor shares are constant.

This paper provides a new approach. The aggregate production function derived here is not Cobb-Douglas: the elasticity of substitution between capital and labor is *endogenous* and is always less than one. Nonetheless, constant factor shares can arise. If the entrepreneurs' productivity distribution is Pareto, the aggregate production function is Cobb-Douglas *in the limit* as unemployment disappears. In this limiting case, factors are paid their marginal product and factor shares are constant. Outside this limit, constant factor shares can coincide with an aggregate production function that is *not* Cobb-Douglas because factors are not necessarily paid their marginal product.

The Cobb-Douglas production function treats labor's share as a mere parameter. Clearly, this approach cannot explain *why* factor shares are stable. Recent papers by Jones (2005) and Lagos (2006) derive the Cobb-Douglas production function from deeper microfoundations, in the spirit of Houthakker (1955). While these papers take very different approaches, each of them uses an underlying distribution of productivity levels or ideas that is Pareto and then aggregates across micro-level production units in order to obtain a Cobb-Douglas aggregate production function.

This paper is most closely related to Jones (2005). In his paper, the "global" production function is asymptotically Cobb-Douglas in the long run as the total number of ideas across the economy grows over time. While the set-up here is quite different,[2] Jones' paper also uses a Pareto distribution of entrepreneurs' productivity levels to derive an aggregate production function that converges to a Cobb-Douglas function. But there is a key difference: labor market frictions are crucial in this paper. The aggregate production function derived here is Cobb-Douglas *only in the limiting case* where these frictions disappear.

Acemoglu (2003) provides microfoundations for the second alternative, in which the long-run stability of factor shares arises from the labor-augmenting nature of technological change. In his paper, the economy converges in the long run to one with purely labor-augmenting technical change and constant factor shares. The approach taken

[2] The differences between my approach and Jones' are discussed in detail in Section 3.

here is different. This paper examines the *shape* of the production function itself, in a way more similar to Jones (2005). While Acemoglu's paper provides an explanation of why technological change may be labor-augmenting, the CES production technology is taken as given. In this paper, the aggregate production function is instead built up directly from microfoundations.

The unified nature of the aggregate production function distinguishes our approach not only from Jones (2005), but also from Lagos (2006). While Jones abstracts from labor market frictions, they are central to Lagos' paper. In his framework, however, the matching and production process, and the wage determination mechanism, are different. Workers and vacancies are randomly matched in bilateral meetings through an *exogenous* matching function. The surplus generated by such matches is divided between workers and vacancies through generalized Nash bargaining in a manner that is standard in Mortensen and Pissarides (1994) style models.

By contrast with Lagos (2006), the model presented here determines wages through a competitive bidding process. When two or more entrepreneurs approach a single worker, there is direct competition to hire the worker. This competitive bidding process is effectively a second-price auction.[3] The entrepreneur with the highest productivity hires the worker and pays a wage equal to the second highest productivity.[4] If exactly one entrepreneur approaches the worker, he is paid a wage equal to his outside option. If no entrepreneurs approach a given worker, he is unemployed.

The relative stability of factor shares emerges through two distinct channels. The first is through the nature of the production technology itself. As the level of entrepreneur entry increases, the aggregate production function approaches a Cobb-Douglas limit. Factors are also paid their marginal product in this limiting case, so factor shares are constant. The second channel is through the wage determination mechanism. Constant factor shares can co-exist with an aggregate production function that is *not* Cobb-Douglas because factors are not necessarily paid their marginal product. In

[3] The process of "bidding" for labor found in Julien et al. (2000) can be seen as a special case of the wage determination mechanism found here, namely where all entrepreneurs (or vacancies) have the same productivity, and one is chosen *randomly* when more than one arrives.

[4] This approach bears some resemblance to Blanchard and Diamond (1994). In their paper, firms that receive multiple job applications from workers "rank" them according to their duration of unemployment and choose the worker who has been unemployed for the shortest period of time. Here, workers that receive multiple job offers from entrepreneurs "rank" them according to their productivity, and choose the entrepreneur with the highest productivity draw.

particular, factor shares are constant in the limit as workers' outside option approaches the minimum value of the entrepreneurs' productivity distribution. In both cases, there is a linear split of output between workers and entrepreneurs that is generated *endogenously*. The parameter determining this linear split is inherited from the underlying distribution of entrepreneurs' productivity levels.

At the same time, the model can account for systematic variation in factor shares – both short-term fluctuations and medium-run changes. First, we might expect labor's share to exhibit medium-run variation due to changes in labor market conditions.[5] The model presented in this paper predicts that labor's share is increasing in workers' outside option, which can be interpreted as either the value of leisure, the value of non-market activity, or simply as unemployment benefits.

Second, it is well-known that labor's share fluctuates in a *counter-cyclical* manner.[6] During a recession, labor's share rises temporarily and capital's share falls. The model can explain this stylized fact. If aggregate productivity shocks are introduced by allowing the underlying distribution of entrepreneurs' productivity to change over time, labor's share is endogenously counter-cyclical provided that the workers' outside option is sufficiently high. Importantly, this counter-cyclical labor share amplifies the effect of productivity shocks on unemployment through the increased incentive for entrepreneur entry during booms and decreased incentive for entry during recessions.

The rest of the paper is structured as follows. Section 2 presents the basic model and derives the aggregate production function for a general distribution of entrepreneur productivity. Section 3 examines special properties of the production function arising from the Pareto distribution. This section also discusses the unified nature of the matching and production process, and the behavior of the elasticity of substitution. Section 4 shows how the equilibrium level of entrepreneur entry is determined. Section 5 demonstrates how equilibrium factor shares are affected by changes in labor market parameters and aggregate productivity shocks. Section 6 concludes. All proofs can be found in the Appendix.

[5]See Blanchard (1997), Caballero and Hammour (1998), and Bentolila and Saint-Paul (2003).

[6]See Young (2004), Gomme and Greenwood (1995), Rios-Rull and Santaeulalia-Llopis (2010).

2 Model

The model is static. There is a continuum of *ex ante* homogeneous risk-neutral potential workers of measure L, indexed by j, and a continuum of risk-neutral entrepreneurs of measure V. The ratio of entrepreneurs to workers is $\theta = V/L$. The equilibrium number of entrepreneurs, given by θ^*, is determined by a free entry condition.

There is an entry cost paid by entrepreneurs, r, which gives each entrepreneur one unit of capital and a single productivity draw from a distribution $G(x)$. Total capital is given by entrepreneurs' demand, $K = V$, and hence $\theta = k = K/L$, the capital/labor ratio. The entry cost can be interpreted as the *rental rate of capital*, since it is the cost of hiring one unit of capital for a single period.

Entrepreneurs are heterogeneous with respect to productivity. The entrepreneurs' productivity distribution, $G(x)$, is continuous and differentiable with support $(x_{\min}, \infty)$ and no mass points. We normalize $x_{\min} = 1$. The *productivity* of an entrepreneur represents his ability to combine labor and capital to produce output. An entrepreneur with productivity x has the ability to produce x units of output using a single unit of capital and a single worker's labor for one period. This ability might be interpreted as a kind of managerial talent.

Entrepreneurs can approach only a single worker. Since workers are homogeneous, entrepreneurs make job offers to workers at random, assigning equal probability to all workers. This gives rise to a Poisson distribution with parameter θ for the number of entrepreneurs approaching each worker.[7] The expected number of entrepreneurs competing for a given worker is θ, so this ratio can be interpreted as a measure of the degree of *competition* for workers' labor.

Workers are aware of the distribution $G(x)$ and the number of entrepreneurs approaching them simultaneously, but are not aware of the productivity levels of particular entrepreneurs. Similarly, entrepreneurs are aware of the number of other entrepreneurs targeting a given worker, but are not aware of their productivity levels.

The process of "matching" workers and entrepreneurs is described below. Each match between a single worker and an entrepreneur with productivity x produces x

[7] Since entrepreneurs approach each worker with equal probability, the Poisson distribution arises because we are taking the limit of a binomial distribution. This urn-ball matching process was first introduced by Butters (1977) and Hall (1979).

units of the final good with price normalized to one. Entrepreneurs receive a payoff of zero if they do not successfully hire a worker. Workers who are not matched receive $z \in [0, 1]$, which is the value of leisure or non-market activity. I refer to the value z as workers' *outside option.*

Matching outcomes

From the worker's perspective, there are three possible matching outcomes: a *multilateral* match, in which more than one entrepreneur approaches that worker; a *bilateral* match, in which exactly one entrepreneur approaches; and *no* match, in which no entrepreneurs approach that worker.

Case 1. *No match.* If no entrepreneurs arrive, the worker is unemployed and output is zero. The worker receives a payoff of z. By the Poisson distribution, this occurs with probability $e^{-\theta}$, so $u(\theta) = e^{-\theta}$ is the *unemployment rate.*

Case 2. *Bilateral match.* If exactly one entrepreneur approaches worker j, he employs the worker and produces output at his own productivity level, x_j. The worker is paid their outside option, z.[8] The entrepreneur's net payoff is $\pi_j = x_j - z - r$.

Case 3. *Multilateral match.* If two or more entrepreneurs approach worker j, they compete for the worker's labor in a second price auction. It is a dominant strategy for each entrepreneur to bid their actual productivity, which is their valuation of the worker's labor. The entrepreneur with the highest productivity level, x_j^1, employs the worker and produces output at productivity x_j^1. The worker's wage equals x_j^2, the second-highest draw, and the entrepreneur's net payoff is $\pi_j = x_j^1 - x_j^2 - r$.

An entrepreneur is *successful* if he has the highest productivity draw for the particular worker he approaches. Successful entrepreneurs always hire the worker, since $x_j > x_{\min} = 1 \geq z$. Expected profits are assumed to be positive for all successful entrepreneurs, i.e. $E_G(x) - z - r > 0$. This assumption is necessary to ensure the existence of an equilibrium level of entrepreneur entry, θ^*.

Assumption 1 *There is positive entry:* $E_G(x) > z + r$

[8] This assumption can be justified by general results from auction theory found in McAfee (1993) and Peters and Severinov (1997). When the number of bidders is determined endogenously by a free entry condition, it is optimal for sellers to set a reserve price equal to their outside option. In this model, it is straightforward to show that workers maximizing their expected payoff would choose to be paid a wage in bilateral matches equal to their outside option. This is verified in Section 4.

Observe that negative profits can occur in both the second and third case. This is because profits depend on the actual productivity draws from $G(x)$, but r is a sunk cost which is paid before the realizations of draws from $G(x)$ are known. Once the cost r has been paid to rent capital, entrepreneurs will always opt to pay wages and produce output because wages are always strictly lower than the value of a worker's output.[9]

2.1 Aggregate production

For any given distribution of *entrepreneur* productivity levels, $G(x)$, the above process gives rise to an endogenous productivity distribution across *workers*, $H_G(x;\theta)$. This new distribution incorporates both the effect of competition between entrepreneurs to hire workers, which leads to an allocation of labor towards higher productivity entrepreneurs, and the impact of labor market frictions, which leads to unemployment for some workers.

Suppose that n entrepreneurs approach a given worker. If $n \geq 1$, the entrepreneur with the highest productivity hires the worker, and the resulting productivity is the maximum of n draws from $G(x)$. This is given by the distribution $G(x)^n$. If no entrepreneurs approach a given worker, he is *unemployed* and produces zero output. Let $H_G(x|n) = G(x)^n$ be the distribution of the worker's productivity conditional on the number of entrepreneurs arriving, n. To obtain the unconditional distribution, $H_G(x;\theta)$, the distribution $H_G(x|n)$ must be weighted by the probability that n entrepreneurs approach, which is given by a Poisson distribution with parameter θ. The distribution $H_G(x;\theta)$ is therefore

$$H_G(x;\theta) = \sum_{n=0}^{\infty} \frac{\theta^n e^{-\theta}}{n!} G(x)^n = e^{-\theta(1-G(x))} \tag{1}$$

The distribution $H_G(x;\theta) = e^{-\theta(1-G(x))}$ has continuous support $[1,\infty)$ and a *mass point* at zero with probability mass $u(\theta) = e^{-\theta}$, the unemployment rate. Since workers are ex ante homogeneous, this is both the distribution of each worker's productivity and the distribution across *all* potential workers.

There is a family of production functions indexed by the underlying distribution

[9] In the second case, this is because $x_j > z$ since $G(x)$ has support $[1,\infty)$ and there are no mass points, so the probability that $x_j = 1$ is zero. In the third case, this is because we have $x_j^1 > x_j^2$, since x_j^1 and x_j^2 are the highest and second-highest draws.

of entrepreneurs' productivity levels, $G(x)$. For any such distribution $G(x)$, output per capita, $y = Y/L$, is the expected value of the workers' productivity distribution, $H_G(x;\theta)$. Since we have $k = \theta$, output per capita can be expressed as a function of k alone.

$$f_G(k) = \int_1^\infty xkg(x)e^{-k(1-G(x))}dx \tag{2}$$

Since this function depends only on the ratio k, the aggregate production function clearly has constant returns to scale in K and L. For any distribution $G(x)$, the intensive production function, $f_G(k)$, is increasing in k and exhibits diminishing marginal returns, but one of the Inada conditions is not satisfied: $f'_G(k) > 0$, $f''_G(k) < 0$, and $\lim_{k\to\infty} f'_G(k) = 0$ as usual, but $\lim_{k\to 0} f'_G(k)$ is finite. (See Appendix A.1.)

These properties of the aggregate production function hold generally for *any* distribution, $G(x)$. In the next section, I consider a particular distribution of entrepreneurs' productivity levels, namely the Pareto distribution. This distribution is used by Jones (2005), Lagos (2006), and Houthakker (1955) to derive Cobb-Douglas aggregate production functions. Jones (2005) argues that empirical studies suggest it is a plausible candidate for representing the distribution of ideas.

3 Aggregate production – the Pareto case

Suppose the underlying distribution of entrepreneurs' productivity levels is the Pareto distribution, $G(x) = 1 - x^{-1/\lambda}$ where $\lambda \in (0, 1)$. In this case, the resulting productivity distribution across workers is $H(x) = e^{-\theta x^{-1/\lambda}}$ with continuous support $[1, \infty)$ plus a mass point at zero. As $e^{-\theta} \to 0$, this distribution converges to the Type II Extreme Value or Fréchet distribution, $H^*(x) = e^{-\theta x^{-1/\lambda}}$ with continuous support $[0, \infty)$.[10]

Calculating the integral in equation (2) when $G(x) = 1 - x^{-1/\lambda}$, and using the fact that $\theta = k$, the intensive form of the production function is

$$f(k) = \gamma(1 - \lambda, \theta)k^\lambda \tag{3}$$

where $\gamma(s, x) = \int_0^x t^{s-1}e^{-t}dt$, the Lower Incomplete Gamma function. (See Appendix

[10] The Fréchet distribution is used in the model of international trade by Eaton and Kortum (2002), and it is derived from microfoundations in Kortum (1997) and Eaton and Kortum (1999). As well as Jones (2005), the Fréchet distribution is also featured in Lucas (2009) and Hsieh et al. (2011).

A.2.) The expression $\gamma(1-\lambda,\theta)$ is increasing in both θ and λ, since $\lambda \in (0,1)$.[11] Importantly, the aggregate production function is *not* Cobb-Douglas, since $\theta = K/L$.

$$Y = \gamma(1-\lambda,\theta)K^{\lambda}L^{1-\lambda} \tag{4}$$

Only in the limit as $e^{-\theta} \to 0$ (i.e. as unemployment goes to zero) do we have $Y = \Gamma(1-\lambda)K^{\lambda}L^{1-\lambda}$, where $\Gamma(s) \equiv \lim_{x\to\infty}\gamma(s,x) = \int_0^{\infty} t^{s-1}e^{-t}dt$, the Gamma function.[12] In this limiting case, the aggregate production function is Cobb-Douglas.

Comparison with Jones (2005)

The fact that the production function is *not* exactly Cobb-Douglas arises from a crucial difference between the aggregation result presented here and that of Jones (2005). He considers a large number of production units, each of which uses a local Leontief production technology. He then takes the convex hull of available ideas to derive a "global" production function, where ideas represent different ways of combining capital and labor to produce output.[13] Since Jones considers the convex hull across the entire economy, and takes the limit as the number of ideas becomes large, he works directly with a Type II Extreme Value or Fréchet distribution. The basic idea is that the production function becomes asymptotically Cobb-Douglas in the long run, as the total number of ideas across the economy grows over time.

In contrast with Jones' approach, we take the highest entrepreneur's productivity draw for each worker and then aggregate across all workers. Since the number of entrepreneurs approaching a given worker during a single time period is relatively small, I consider the *exact* distribution which arises for finite θ, namely $H(x) = e^{-\theta x^{-1/\lambda}}$ with continuous support $[1,\infty)$ plus a mass point at zero. The "gap" in this distribution corresponds to frictional unemployment, which occurs with probability $u(\theta) = e^{-\theta}$. While Jones' global production function becomes Cobb-Douglas over time as the number of ideas grows large, the aggregate production function derived here becomes Cobb-Douglas only as the labor market frictions disappear.

[11] The Lower Incomplete Gamma function is *decreasing* in s for any $s \in (0,1)$.

[12] The Gamma function is the unique function that extends the factorial function to $\mathbb{R}^+$.

[13] See also Caselli and Coleman (2006), who use a related approach to examine technology choice and the world technology frontier.

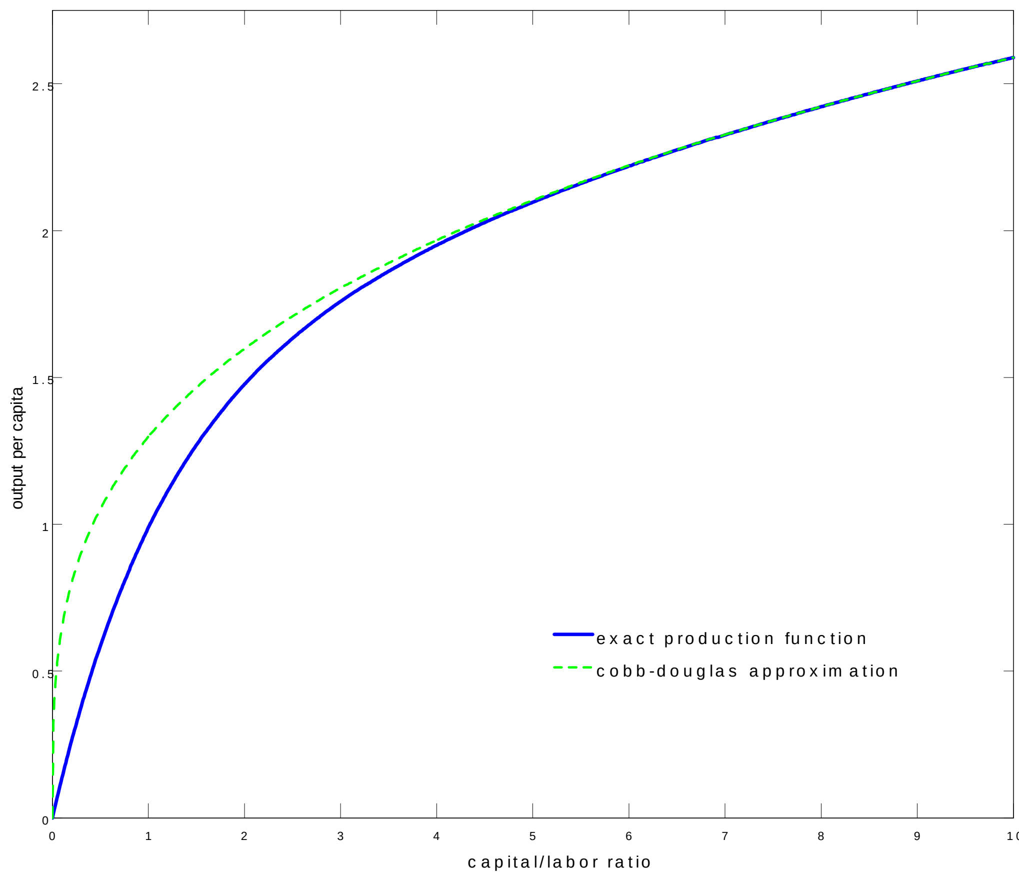

Figure 1 Comparison with Cobb-Douglas production function

Figure 1 shows how the exact production function (for $\lambda = 0.3$) converges to a Cobb-Douglas function as $e^{-\theta} \to 0$. The graph suggests that the Cobb-Douglas production function might be an appropriate reduced-form approximation in applied work, provided that θ is sufficiently high. For theoretical purposes, however, the exact production function developed in this paper enables us to explain systematic movements in factor shares that are ruled out by simply assuming a Cobb-Douglas production function at the outset. I turn to this objective in Section 5.

3.1 Unified matching and production function

Search-theoretic models of the labor market generally feature an exogenous matching function, $M(U, V)$, which gives the number "matches", or the number of employed workers, as a function of the number of unemployed workers and job vacancies. It is commonly assumed that the matching function is constant returns to scale, so $M(U, V)/U = m(\theta)$, where $\theta = V/U$.[14] In such models, aggregate output is simply output per match multiplied by the number of matches. The framework developed here is different. The aggregate production function, $Y = \gamma(1-\lambda, \theta)K^{\lambda}L^{1-\lambda}$, is *not* just a production function multiplied by a matching function. Instead, it is *unified* in the sense that it incorporates both a matching and a production process.

Matching and production

By a *matching* process, I mean simply the process by which workers and entrepreneurs are "matched" and the unemployment rate is determined. By a *production* process, I mean the process according to which a given match produces a certain quantity of output. In this model, matching and production are two aspects of a single process. If no entrepreneurs approach a given worker, he is unemployed. If at least one entrepreneur approaches, the entrepreneur with the highest productivity wins the right to hire to worker and produce output at that productivity level.

The production process incorporates the fact that an important effect of the *competition* between entrepreneurs to hire workers is an allocation of labor towards more productive entrepreneurs. A greater number of entrepreneurs competing for a worker's labor implies a higher expected value of the maximum productivity, which leads to higher output for that worker and hence higher aggregate labor productivity. Seen in this light, the expression $\gamma(1-\lambda, \theta)$ can be interpreted as a kind of *generalization* of the matching function which includes the effect of competition between entrepreneurs on aggregate productivity.

To understand how this unified production function works, we can isolate both the matching function and the production function by considering two limiting directions. Recall that L represents *all* workers, including the unemployed. The first direction gives us the *matching* function (i.e. the number of "matches" or employed workers).

[14] See Petrongolo and Pissarides (2001) and Rogerson et al. (2005) for surveys.

Consider the limit as $\lambda \to 0$. This is equivalent to assuming a degenerate underlying distribution, $G(x)$. Since $\gamma(1,\theta) = 1 - e^{-\theta}$, the aggregate production function collapses to the urn-ball matching function, $m(\theta) = 1 - e^{-\theta}$, first introduced by Butters (1977) and Hall (1979).[15]

The second direction gives us the limiting *production* function (i.e. output per employed worker). If we take the limit as $e^{-\theta} \to 0$, we obtain the Cobb-Douglas production function, $Y = \Gamma(1-\lambda)K^{\lambda}L^{1-\lambda}$. In this limiting case, there is no unemployment and hence L is the number of employed workers. The expression $\Gamma(1-\lambda)$ is the maximum level for $\gamma(1-\lambda,\theta)$ as the number of entrepreneurs goes to infinity and the labor market frictions disappear.

Unemployment and capital utilization

The unified nature of the matching and production process affects the way in which the marginal products of labor and capital should be interpreted. Importantly, the marginal product of labor represents the effect on aggregate output of an extra *potential* worker, who may end up either employed or unemployed, depending on matching outcomes. In a similar manner, the marginal product of capital represents the marginal contribution of an extra unit of *hired* capital, which may end up either "utilized" or unutilized, depending on the outcome of the entrepreneurs' competitive bidding process.

It would be a mistake to think of unemployed workers as excluded from contributing to aggregate output, since the existence of such workers increases the level of entrepreneur entry, θ, which in turn leads not only to a higher number of matches but also to an increase in aggregate labor productivity through the productivity process described above. Similarly, it would be wrong to think of the capital hired by unsuccessful entrepreneurs as being simply *idle*, in the sense of making no contribution to aggregate output. While there is a sense in which this capital is indeed "unutilized", it is certainly not *unproductive*, since it is the competition between entrepreneurs, each of whom rent capital, that drives the increase in aggregate labor productivity.

Consider the following analogy. Suppose there is a worker who has at his disposal a number of different kinds of capital. For concreteness, imagine that he has a saw, a

[15] The urn-ball matching function arises endogenously in directed search models of the labor market as the economy becomes large. For papers related to directed search see, for example, Peters (1991), Montgomery (1991), Burdett et al. (2001), Julien et al. (2000), Shi (2001), Shimer (2005), Albrecht et al. (2006), and Galenianos and Kircher (2009).

hammer, and a chisel available for use. At any given time, suppose he chooses to use the best implement at hand, i.e. the one that makes his labor the most productive. Now imagine that he chooses the hammer. While the saw and the chisel are not "utilized" during this period, it would be wrong to say that they are unproductive in the sense of failing to contribute to the worker's output in this period. For the worker *could* have used any one of these implements, and it is the process of selecting the best available implement for any given task that increases the worker's productivity.

The unified production and matching process described here captures – in a stark and simple manner – the idea that all potential workers and all units of hired capital contribute to aggregate output, regardless of whether these workers turn out to be employed and regardless of whether these entrepreneurs are successfully matched.

3.2 Endogenous elasticity of substitution

One important property of this aggregate production function is that the elasticity of substitution between capital and labor, σ, is *not* equal to one. In fact, it is not constant at all but varies with $\theta = k$. Starting with the definition of σ in Arrow et al. (1961),

$$\sigma = \frac{-f'(k)(f(k) - kf'(k))}{kf(k)f''(k)} \tag{5}$$

In the appendix, I prove that the elasticity of substitution is given by

$$\sigma = \frac{\lambda + \varepsilon(1-\lambda, \theta)}{\lambda + \varepsilon(2-\lambda, \theta)} \tag{6}$$

where $\varepsilon(s, x)$ is the elasticity of $\gamma(s, x)$ with respect to x, namely $\varepsilon(s, x) = \frac{x^s e^{-x}}{\gamma(s,x)}$. (See Appendix A.3 for proof.) To interpret the term $\varepsilon(s, x)$, recall that $\gamma(1-\lambda, \theta)$ can be seen as a kind of generalization of the urn-ball matching function which includes the effect of competition between entrepreneurs on aggregate productivity. Thus the term $\varepsilon(1-\lambda, \theta)$ can be seen as a generalization of the elasticity of the matching function with respect to entrepreneurs.

In the limit as $\theta \to 0$, we have $\sigma \to 1/2$ since $\lim_{x\to 0} \varepsilon(s, x) = s$. In the limit as $\theta \to \infty$, the elasticity of substitution $\sigma \to 1$ as $\lim_{x\to\infty} \varepsilon(s, x) = 0$. This limit of one is approached from below. The elasticity of substitution between capital and labor, σ, is always less than one. (See Appendix A.4 for proof.) Importantly, the fact that

$\sigma < 1$ is a theoretical result here, not an assumption. This contrasts with the standard alternative to assuming a Cobb-Douglas aggregate production function in applied work - namely, to start with a CES production function and assume a particular value of σ, as estimated by empirical studies. Here, the elasticity of substitution is *endogenous*.

The question of the exact value of the elasticity σ is open to debate. Estimates vary widely, from 0.3 to approximately one, but most empirical studies indicate that the elasticity is significantly below one.[16] To get a rough idea of the sort of numbers predicted by the model, suppose that $\lambda < 0.5$ and θ is between 2.5 and 3.5. This corresponds to unemployment rates between around 3% and 8%. For this range, σ varies between 0.28 and 0.72. For example, if $\lambda = 0.3$ and $k = \theta = 3$ (i.e. an unemployment rate of around 5%), we have $\sigma = 0.54$.

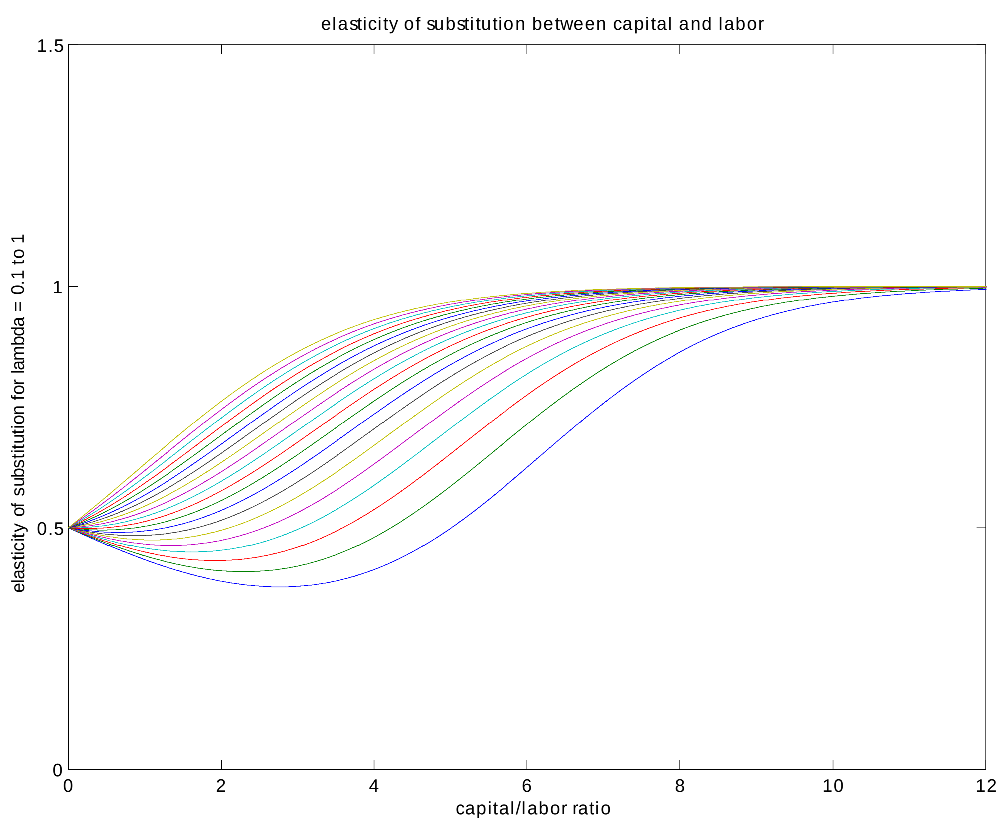

Figure 2 Elasticity of substitution between capital and labor

[16]See footnote 3 in Acemoglu (2003) for a summary of the empirical estimates of this elasticity. Also see Antras (2004).

Figure 2 illustrates the elasticity of substitution, σ, as a function of $k = \theta$ for different values of λ. Clearly, σ is *not* monotonic in k for all values of λ. This contrasts with the variable elasticity of substitution (VES) class of production functions introduced by Revankar (1971), which include the constant elasticity of substitution (CES) production functions developed in Arrow et al. (1961) as special cases.[17] In this paper, the relationship between σ and k is a more subtle one that arises endogenously from micro-foundations.

Until now, the elasticity σ has been considered simply as a function of k and λ, not the equilibrium $k^* = \theta^*$, which itself depends on λ. In the next section, I turn to the issue of determining the equilibrium level of entrepreneur entry, θ^*.

4 Equilibrium

The equilibrium level of entrepreneur entry, θ^*, is determined by a free entry condition which ensures the expected payoff for entrepreneurs, net of entry cost, is zero. The expected net payoff for entrepreneurs, $\pi(\theta)$, is determined by both the probability of being *successful* (i.e. having the highest productivity for a given worker) and the expected payoff for a successful entrepreneur. The former is determined by the Poisson arrival process, while the latter depends on the wage determination mechanism.

Recall that wages are determined as follows. After an entrepreneur approaches a given worker, there are two possibilities. Either the entrepreneur is alone, or there is competition for the worker. If the entrepreneur is alone in approaching a worker, he employs that worker and pays him the worker's outside option, z. If there are one or more other entrepreneurs approaching that worker, the entrepreneur with the highest productivity hires the worker at a wage equal to the next highest productivity. The expected payoff for a successful entrepreneur is the output for a single worker (given by the entrepreneur's productivity), minus the wage paid.

[17] Just as the CES production function was derived by setting σ to be constant and then solving the resulting differential equation, the VES production function was obtained by setting σ to be a linear function of k and then solving for the production function. Clearly, then, σ is monotonic in k. See Revankar (1971) for details.

Free entry condition

This environment is equivalent to a second-price auction where each entrepreneur's valuation of a worker's labor equals their productivity draw from the distribution $G(x)$. There is a *stochastic* number of bidders determined by the Poisson distribution.[18] In Appendix A.5, I derive the following free entry condition:

$$\pi(\theta) = \int_1^\infty e^{-\theta(1-G(x))}(1-G(x))dx + e^{-\theta}(1-z) - r = 0 \tag{7}$$

To ensure the existence of an equilibrium θ^*, it is necessary to use the assumption that $E_G(x) > r + z$. It is easy to see that $\pi'(\theta) < 0$, so if there exists a θ such that $\pi(\theta) > 0$, the equilibrium θ^* must be unique. Since $\pi(0) = E_G(x) - z - r$, the assumption that $E_G(x) > r + z$ ensures that $\pi(0) > 0$. So there exists a unique equilibrium level of entrepreneur entry, θ^*, such that $\pi(\theta^*) = 0$. (See Appendix A.6.)

Proposition 1 *There exists a unique equilibrium level of entrepreneur entry,* θ^**.*

For any distribution $G(x)$, the equilibrium θ^* is decreasing in workers' outside option, z, and decreasing in the rental rate of capital, r. This implies that the equilibrium unemployment rate, $u(\theta^*) = e^{-\theta^*}$, is increasing in workers' outside option and increasing in r. These results are intuitive. A higher workers' outside option, z, means that entrepreneurs are deterred by the lower expected profits, so θ^* is lower and unemployment is higher. A lower rental rate rate of capital, r, implies higher expected profits for entrepreneurs, which leads to an increase in θ^* and a lower unemployment rate. (See Appendix A.7.)

Wage dispersion

The equilibrium level of entrepreneur entry, θ^*, determines *expected* wages, $w(\theta^*) = f(\theta^*) - r\theta^*$. This is just total wages divided by the number of *potential* workers, i.e. the expected payoff from market activity for all workers, including the unemployed. While expected wages are pinned down by the equilibrium θ^*, the wage determination mechanism gives rise to residual wage dispersion across workers. This is because the actual wage paid to a particular worker depends on the productivity levels of the entrepreneurs bidding for that worker. In fact, even when workers turn out to be identical

[18] This environment is similar to the competing auctions framework analysed by Peters and Severinov (1997). Auctions with a stochastic number of bidders were first studied by McAfee and McMillan (1987).

with respect to productivity *ex post* (i.e. the highest entrepreneur's productivity is the same for both workers), there will be residual wage dispersion. Two different workers with identical productivity outcomes may face a different profit/wages split between workers and entrepreneurs simply because this split depends on the value of the *second* highest productivity draw.

The expected payoff for workers from both market *and* non-market activity is $\hat{w}(\theta^*) = f(\theta^*) - r\theta^* + ze^{-\theta}$. We can now verify that if workers were able to choose *ex ante* a bilateral wage, b^*, in order to maximize their expected payoff, $\hat{w}(\theta^*)$, they would choose b^* equal to their outside option, z. Suppose that workers anticipate the effect of b^* on equilibrium entrepreneur entry, θ^*, and solve the following problem:

$$b^* = \arg\max \; f(\theta^*(b)) - r\theta^*(b) + ze^{-\theta(b)} \tag{8}$$

The unique solution to the workers' optimization problem is $b^* = z$.[19] This justifies our initial assumption that workers are paid their outside option in bilateral matches.

Pareto distribution

Now suppose that $G(x)$ is Pareto, $G(x) = 1 - x^{-1/\lambda}$ where $\lambda \in (0, 1)$. In Appendix A.8, I show that the equilibrium ratio θ^* is the unique solution to

$$\pi(\theta) = \lambda\theta^{\lambda-1}\gamma(1-\lambda, \theta) + (1-z)e^{-\theta} - r = 0 \tag{9}$$

Factors are paid their marginal product, $r = MPK$ and $w = MPL$, if and only if either $z = 0$ or $e^{-\theta} \to 0$. The marginal product of capital is $MPK = f'(\theta) = \lambda\theta^{\lambda-1}\gamma(1-\lambda, \theta) + e^{-\theta}$, so $r = MPK$ only if either $z = 0$ or $e^{-\theta} \to 0$. Expected wages are given by $w(\theta) = f(\theta) - r\theta = (1-\lambda)y - (1-z)\theta e^{-\theta}$, while the marginal product of labor is $MPL = f(\theta) - \theta f'(\theta) = (1-\lambda)y - \theta e^{-\theta}$. This is the marginal product of an extra *potential* worker, as discussed in Section 3.1. If $z > 0$, workers are paid more than their marginal product. The "wage gap", $w - MPL = z\theta e^{-\theta}$, disappears only

[19] The first order condition is:

$$\frac{d\theta^*}{db}(f'(\theta) - r - ze^{-\theta}) = 0$$

Now $\frac{d\theta^*}{db} < 0$, so the first order condition holds if and only if $f'(\theta) - r = ze^{-\theta}$. At the same time, we know from Appendix A.1 that $f'(\theta) = \int_1^\infty e^{-\theta(1-G(x))}(1-G(x))dx + e^{-\theta}$. If workers are paid b in bilateral matches, the free entry condition implies that $f'(\theta) - r = be^{-\theta}$ for the equilibrium θ^*. Hence $b^* = z$.

when workers' outside option is zero, or in the frictionless limit as $e^{-\theta} \to 0$.

The expected payoffs for both workers and successful entrepreneurs are approximately *linear* in output per capita when the distribution $G(x)$ is Pareto. Both the fact that the profit/wages split is linear, and the particular value of the linear split, are endogenous here. In the limit as $e^{-\theta} \to 0$, expected wages are $w = (1-\lambda)y$ and the expected payoff for a successful researcher is $r_s = r\theta = \lambda y$. The parameter λ can be interpreted roughly as a kind of "shadow" bargaining parameter, in the sense that it pins down the profit/wages split. This linear division of income at the aggregate level occurs simultaneously with the micro-level wage dispersion across individual workers and the wide variation in profits across entrepreneurs.

The equilibrium θ^* is increasing in λ, and the equilibrium unemployment rate, $u(\theta^*)$, is decreasing in λ. Output per capita, $f(\theta) = \theta^{\lambda}\gamma(1-\lambda,\theta)$, is increasing in λ. (See Appendix A.9 for proof.) An exogenous change in the parameter λ can be interpreted as an *aggregate productivity shock* because the expected value of a productivity draw x from $G(x)$ is $1/(1-\lambda)$, which is increasing in λ. By contrast, *idiosyncratic* productivity shocks are the joint effect of two random variables: the number of entrepreneurs approaching a given worker, plus the realizations of draws from $G(x)$. Effectively, idiosyncratic productivity shocks correspond to draws from the distribution $H(x) = e^{-\theta x^{-1/\lambda}}$, which jointly captures both of these. The mean of this distribution of idiosyncratic productivity shocks is increasing in λ.

Constrained efficiency

An economy is called *constrained efficient* if the socially optimal ratio, θ_P, equals the decentralized equilibrium outcome, θ^*. The socially optimal θ is the value chosen by a social planner who faces the same coordination frictions found in the decentralized economy – that is, the same process of "matching" between entrepreneurs and workers – but is free to choose the ratio θ_P directly. The social planner's objective is to maximize total output plus the value of non-market activity, minus the total costs of entry. That is, θ_P solves the following:

$$\theta_P = \arg\max_{\theta \geq 0} \; f(\theta) + ze^{-\theta} - r\theta \tag{10}$$

For any distribution $G(x)$, the economy satisfies $\theta_P = \theta^*$.[20] (See Appendix A.10 for proof.) However, this result needs to be interpreted with caution. In this model, the notion of constrained efficiency is problematic. The "entry costs" here are not simply costs of vacancy creation, as they would be in a standard search-theoretic model of the labor market. Instead, these costs represent the rent paid to owners of capital. If we were to include capital's share of income, the social planner's objective would be to maximize $f(\theta) + ze^{-\theta}$. This problem has no solution: it is optimal simply to choose θ to be as large as possible.

Nonetheless, the Pareto case enables us to readily see a connection between the above efficiency result and the Hosios rule found in the search-theoretic literature (Hosios (1990)). This rule states that the entry of vacancies is constrained efficient when the bargaining parameter of vacancies is equal to the elasticity of the *matching* function with respect to vacancies. Bearing in mind the above caveat, the result obtained here can be reconciled with the Hosios rule by considering the following example.

Imagine that r is purely a cost, so that $\theta_P = \theta^*$ does truly represent constrained efficiency. Suppose that workers bargain with *successful* entrepreneurs using generalized Nash bargaining, where entrepreneurs' bargaining parameter is β and workers' outside option is zero. Consider the zero unemployment limit as $e^{-\theta} \to 0$. The expected payoffs for workers and successful entrepreneurs in the second-price auction, $(1-\lambda)y$ and λy respectively, correspond to those achieved through Nash bargaining using a specific bargaining parameter, $\beta = \lambda$. Since constrained efficiency holds in the second-price auction framework, it must also hold in the Nash bargaining environment if $\beta = \lambda$, where λ is the elasticity of the aggregate *production* function with respect to entrepreneurs.

5 Factor income shares

In this section, I provide expressions for factor income shares and show how they vary with an arbitrary level of entrepreneur entry, θ. In the next section, I consider how *equilibrium* factor shares respond to exogenous changes in the parameters λ, z, and r.

[20] In directed search models, constrained efficiency generally does obtain in large markets. In a sense, the above proposition generalizes some of the existing results on constrained efficiency found in directed search models where vacancies compete for workers' labor (eg. Julien et al. (2000)). To see this, consider the special case where entrepreneurs are just vacancies – i.e. where the entrepreneurs' productivity distribution $G(x)$ is degenerate and r is the cost of vacancy creation.

Imagine there are owners of capital who are paid the cost r by entrepreneurs to lend them a single unit of capital for one period. Since there is a zero profit condition for entrepreneurs, output is simply split between workers and owners of capital. For any given distribution $G(x)$, the share of income going to capital, $s_K(G)$, is rK/Y. This is the *actual* share of output going to capital, not the standard $f'(k).K/Y$, since factors are not necessarily paid their marginal product. Capital's share of income is

$$s_K(G) = \frac{\displaystyle\int_1^\infty e^{-\theta(1-G(x))}\,(1-G(x))\,dx + e^{-\theta}(1-z)}{\displaystyle\int_1^\infty e^{-\theta(1-G(x))} x g(x) dx} \tag{11}$$

The significance of workers' outside option, z, decreases as $e^{-\theta} \to 0$. The intuition here is simple. As θ becomes small (i.e. unemployment is high), workers' outside option is increasingly important, since wages equal z if exactly one entrepreneur arrives. The importance of workers' outside option, z, is negligible if θ is very high (i.e. unemployment is low) since the probability of having only one entrepreneur approach a given worker is very small. Once two or more entrepreneurs *compete* for a particular worker, wages are bid up and workers' outside option is immaterial.

If $G(x)$ is Pareto, the capital share of income, s_K, simplifies to

$$s_K = \lambda + (1-z)\varepsilon(1-\lambda, \theta) \tag{12}$$

where $\varepsilon(s, x)$ is the elasticity of $\gamma(s, x)$ with regard to x.[21] If $z = 1$, factor shares are independent of the value of θ. If $z < 1$, factor shares vary depending on θ. In the limit as $e^{-\theta} \to \infty$, we have $\gamma(s, x) \to \Gamma(1-\lambda)$, so the elasticity $\varepsilon(s, x)$ goes to zero.

To interpret this, consider the special case where $z = 0$. In this case, factors are paid their marginal product. Capital's share is equal to the elasticity of the aggregate production function, $Y = \gamma(1-\lambda, \theta)K^\lambda L^{1-\lambda}$, with respect to capital, namely $s_K = \lambda + \varepsilon(1-\lambda, \theta)$. Dissecting this further, capital is paid both the elasticity, λ, of the Cobb-Douglas production function, $Y = K^\lambda L^{1-\lambda}$, plus the elasticity, $\varepsilon(1-\lambda, \theta)$, of the generalized matching function, $\gamma(1-\lambda, \theta)$, with respect to capital. In the degenerate case where $\lambda = 0$, capital is paid a share of output equal to the elasticity of the matching

[21] The constancy of the first term, λ, arises from the fact that $(1 - G(x))/g(x) = \lambda x$. The Pareto distribution is the *only* distribution with this property.

function, $m(\theta) = 1 - e^{-\theta}$, with respect to θ. More generally, when $\lambda > 0$, capital is paid for its contribution to both matching *and* production.

Proposition 2 *Factor shares are constant, $s_K = \lambda$ and $s_L = 1 - \lambda$, in two cases:*
(i) in the limit as $e^{-\theta} \to 0$ (i.e. as the unemployment rate goes to zero);
(ii) in the limit as $z \to 1$ (i.e. as workers' outside option approaches $x_{\min} = 1$).

To be precise, by *constant* factor shares I mean simply that factor shares are constant provided the underlying productivity distribution, $G(x)$, is held *fixed*. For the Pareto distribution, this means that factor shares are constant if they depend *only* on λ.[22] To get a sense of what the limit as $z \to 1$ means, let $p = E_G(x) = 1/(1-\lambda)$. If $z = 1$, this is equivalent to $z = (1-\lambda)p$. For example, if $\lambda = 0.3$, this means that workers' outside option is equal to 70% of the average entrepreneur's productivity.

In general, if factors are paid their marginal product, factor shares are constant (due to the production technology alone) if and only if the aggregate production function is Cobb-Douglas. Equivalently, if we have constant factor shares, then factors are paid their marginal product if and only if the aggregate production function is Cobb-Douglas.[23] Both of the above cases are consistent with Proposition 2, but in different ways. In case (i), factors are paid their marginal product *and* we have a Cobb-Douglas aggregate production function, while in case (ii), factors are *not* paid their marginal product and the aggregate production function is *not* Cobb-Douglas.

If we consider the direct effect of θ on factor shares *outside* of equilibrium, then capital's share is decreasing in θ. This is because the term $\varepsilon(1-\lambda, \theta)$ is decreasing in θ. (See Appendix A.11 for proof.) Equivalently, labor's share is increasing in θ. This result is intuitive since θ is a measure of the degree of *competition* for workers' labor. Greater competition for workers results in a higher labor share, while less competition for workers leads to a lower labor share.

As the number of entrepreneurs goes to infinity, $e^{-\theta} \to 0$, the unemployment rate goes to zero, and the aggregate production function approaches its Cobb-Douglas limit.

[22] In the next section, I allow the distribution $G(x)$ to change over time and interpret changes in the parameter λ as a series of aggregate productivity shocks. In this section, however, the distribution $G(x)$ is treated as fixed.

[23] Here "constancy of factor shares" means simply constancy that is due to the nature of the aggregate production function alone. The possibility of factor shares that are asymptotically constant along a balanced growth path, due to labor-augmenting technical change and a CES production function with $\sigma < 1$, is a separate question (see Acemoglu (2003)).

In this limiting case, capital's share approaches its lower bound, $s_K = \lambda$. As the number of entrepreneurs approaches zero, unemployment is pervasive and capital's share reaches its upper bound, $s_K = 1 - z(1 - \lambda)$. The degree of variability of factor shares across different values of θ is decreasing in workers' outside option, z. The importance of θ diminishes in the limit as $z \to 1$, where there is *no* dependence on θ. At the other extreme, if $z = 0$, there is maximal variation in factor shares.

5.1 Equilibrium factor shares

I now consider how *equilibrium* factor shares vary depending on labor market conditions and aggregate productivity shocks, as represented by shifts in the key parameters λ and z. As we will see, there are both direct effects and indirect effects via the channel of entrepreneur entry, θ^*.

The equilibrium capital share is $s_K^* = \lambda + (1 - z)\varepsilon(1 - \lambda, \theta^*)$ where θ^* solves (9), the free entry condition. First, the equilibrium capital share, s_K^*, is increasing in the rental rate of capital, r. The result is obvious in this case, since there is only the indirect effect on s_K^* through θ. If r increases, θ^* decreases, and since $\varepsilon(1 - \lambda, \theta)$ is decreasing in θ, the effect is an increase in the equilibrium capital share s_K^*.

Effect of workers' outside option

Next, consider how s_K^* varies with the parameter z. If z increases, the direct effect is that s_K should decrease. However, the indirect effect is that an increase in z leads to a lower level of equilibrium entrepreneur entry, θ^*. Since $\varepsilon(1 - \lambda, \theta)$ is decreasing in θ, this indirect effect should lead s_K^* to increase. The net result is not obvious. It turns out that the direct effect dominates: the equilibrium capital share is decreasing in workers' outside option, z. While an increase in z increases unemployment, it also increases labor's share of output. (See Appendix A.12 for proof.)

Proposition 3 *The equilibrium labor share, s_L^*, is increasing in workers' outside option, z.*

Given that the equilibrium labor share is increasing in workers' outside option, z, we might wonder whether expected wages are also increasing in z. Surprisingly, it turns out that expected wages are actually decreasing in z. Recall that expected wages are

$w(\theta^*) = f(\theta^*) - r\theta^*$.[24] Alternatively, $w(\theta) = s_L^* f(\theta)$. We know from Proposition 3 that s_L^* is increasing in z. However, $f(\theta)$ is decreasing in z, since it is increasing in θ and $\theta'(z) < 0$. So there are two opposing effects: an increase in z leads to an increase in labor's *share* of output, but it also decreases the *level* of output due to the lower entry of entrepreneurs, which has a negative effect on aggregate productivity. In fact, the depressing effect on productivity always dominates, so expected wages are decreasing in z.[25] (See Appendix A.13 for proof.)

Proposition 4 *Expected wages, $w(\theta^*)$, are decreasing in workers' outside option, z.*

This result may seem counter-intuitive. However, if we bear in mind the various matching possibilities, and their different effects on wages, the result makes more sense. Let w_n be the expected wage that results when n entrepreneurs approach a given worker.[26] We can express expected wages as $w(\theta) = \Pr(n = 0)w_0 + \Pr(n = 1)w_1 + \Pr(n \geq 2)w_2$. Since we are including market activity only, $w_0 = 0$.

$$w(\theta) = \theta e^{-\theta} z + (1 - e^{-\theta} - \theta e^{-\theta})w_2 \tag{13}$$

Clearly, $w_1 = z$ is increasing in z, and the equilibrium probability $\Pr(n = 1) = \theta e^{-\theta}$ is also increasing in z if $\theta > 1$. However, the probability $\Pr(n \geq 2) = 1 - e^{-\theta} - \theta e^{-\theta}$ is decreasing in z. The wage for multilateral matches, w_2, is also decreasing in z, since it is increasing in θ. The above result implies that the negative effect on both the probability of a multilateral match and the wages obtained in such matches leads to the overall negative impact of z on expected wages. Higher z means higher wages for workers in bilateral matches, but it also leads to lower entrepreneur entry and less competition for workers, which decreases both the relative proportion of multilateral matches and the wages paid in these matches.[27]

[24] This is just total wages divided by the number of *potential* workers, ie. it is the expected payoff from market activity for all workers, including the unemployed. It excludes the payoff for unemployed workers from non-market activity, z.

[25] This result holds for expected wages across all *potential* workers, but not necessarily if we restrict attention to employed workers. If we were to consider expected wages conditional on employment, the result is ambiguous.

[26] Here w_2 means the expected wage when two or more entrepreneurs approach a worker.

[27] The wage for a multilateral match is always strictly greater than workers' outside option, i.e. $w_2 \geq z$, since $w_2 > 1$ and $z \leq 1$.

Aggregate productivity shocks

Next, I consider how equilibrium factor shares respond to aggregate productivity shocks. To do this, we allow the underlying entrepreneurs' productivity distribution, $G(x)$, to change over time. Changes in the distribution $G(x)$ occur through shifts in the mean of this distribution. For the Pareto distribution, a positive aggregate productivity shock is represented by an increase in $p = E_G(x) = 1/(1-\lambda)$, or equivalently an increase in the parameter λ.[28] The average mean (across time) of the distribution $G(x)$ is given by $\bar{p}$ and the average value of the parameter λ is denoted by $\bar{\lambda}$ (i.e. there is no trend).

To fix terminology, I take *procyclical* to mean "increasing in a positive aggregate productivity shock". In the limiting cases where factor shares are "constant" (i.e. depend only on λ), capital's share is clearly procyclical. If either $e^{-\theta} \to 0$ or $z \to 1$, capital's share is $s_K^* = \lambda$. We have $ds_K^*/dp = (1-\lambda)^2 > 0$ and the elasticity of capital's share with respect to p is $\varepsilon(s_K^*, p) = (1-\lambda)/\lambda$. Capital's share is procyclical, although factor shares still exhibit long-run stability in the sense that factor shares are stationary. In periods when λ increases, capital's share rises temporarily, while during periods when λ decreases, capital's share falls.[29] Overall, however, factor shares are stationary.

Outside of these special cases, the effect of an aggregate productivity shock is more complicated. Recall that the equilibrium capital share is $s_K^* = \lambda + (1-z)\varepsilon(1-\lambda, \theta^*)$. There are two distinct channels through which changes in λ may affect equilibrium factor shares. When there is an increase in λ, there is both a simple direct effect through the first term, λ, and a more complex effect through the term $\varepsilon(1-\lambda, \theta)$. The direct effect implies that s_K^* should be *increasing* in λ, the "shadow" bargaining parameter for entrepreneurs. Since the entrepreneurs' shadow bargaining parameter is endogenously equal to the parameter λ governing the mean of the underlying distribution, it naturally increases when there is a positive aggregate shock.

There is also a more complicated effect through the term $\varepsilon(1-\lambda, \theta)$. This term captures the impact of the competition between entrepreneurs to hire workers. There are two components to this channel. First, there is an indirect effect whereby θ^* increases with λ, since more entrepreneurs are attracted by the higher expected profits.

[28]For the Pareto distribution, an increase in λ also increases the variance, so this is not purely an increase in the mean. This is one limitation of using a Pareto distribution.

[29]It is important to emphasize that this is *not* simply a direct shock to capital share, since in this model it is a *result* that $s_K = \lambda$, the parameter of the underlying distribution, as $e^{-\theta} \to 0$ or $z \to 1$.

This leads to greater competition for workers' labor, which has the effect of decreasing s_K^*, since $\varepsilon(1-\lambda,\theta)$ is decreasing in θ (for a given λ). Second, there is a direct effect through the function $\varepsilon(1-\lambda,\theta)$. Since $\varepsilon(1-\lambda,\theta)$ is decreasing in λ (for a given θ), an increase in λ also has the effect of decreasing s_K^*. The overall effect of an increase in λ through the term $\varepsilon(1-\lambda,\theta)$ is therefore one of *decreasing* s_K^*.

It turns out that either the positive or the negative channel may dominate. Workers' outside option is key in determining the relative importance of these two channels. The term $(1-z)$ acts as a kind of "weight" on the component $\varepsilon(1-\lambda,\theta)$ in capital's share. A higher z gives less weight to this component and more weight to the entrepreneurs' shadow bargaining parameter, λ. In the appendix, I provide a sufficient condition on z for the equilibrium capital share to be increasing in λ. (See Appendix A.14 for proof.)

Proposition 5 *The equilibrium capital share, s_K^*, is procyclical provided that workers' outside option, z, is sufficiently high:*

$$z > \frac{1}{2-\lambda}$$

The model predicts that capital's share is procyclical provided that $z > 1/(2-\lambda)$.[30] If $\lambda < 0.5$, then $z > 2/3$ will suffice. For example, if $\bar{\lambda} = 0.3$, then $z > 0.5\bar{p}$ will suffice. If workers' outside option is sufficiently high, the model is therefore able to capture the stylized fact that labor's share fluctuates in a counter-cyclical manner. The fact that labor's share is *endogenously counter-cyclical*, rather than being determined by a fixed parameter such as workers' bargaining power, suggests that the effect of productivity shocks on entrepreneur entry, and thereby on unemployment, will be amplified. Entry becomes even more attractive for entrepreneurs when there is a positive aggregate productivity shock, and even less attractive when there is a negative productivity shock.

[30] By Assumption 1, we require that $E_G(x) > z+r$. This is necessary in order to ensure the existence of an equilibrium θ^*. For the Pareto distribution $G(x)$, the sufficient condition in Proposition 1.5 is consistent with Assumption 1 if and only if the following holds:

$$r < \frac{1}{(1-\lambda)(2-\lambda)}$$

Since the right hand side is increasing in λ for $\lambda \in [0,1]$, the assumption that $r < 1/2$ will suffice.

6 Conclusion

This paper develops microfoundations for a *unified* aggregate production function which features a built-in matching function that determines the unemployment rate. I apply this unified framework to the problem of explaining the behavior of factor shares. Factor shares are constant in distinct two cases: *(i)* as the level of entrepreneur entry becomes large and the aggregate production function approaches its Cobb-Douglas limit; *(ii)* as workers' outside option approaches the minimum value of the entrepreneur's productivity distribution. In the latter case, factor shares may be constant even though the aggregate production function is *not* Cobb-Douglas.

The model can also potentially account for systematic variation in factor shares both over time and across countries. On the one hand, the model predicts that countries with higher values of leisure or non-market activity, or higher unemployment benefits, have a higher labor share and higher unemployment rates. On the other hand, the model can explain the fact that labor's share is *counter-cyclical*: during a recession, labor's share rises temporarily while the profit share falls. One potential implication of the endogenously counter-cyclical labor share is the amplification of the effects of productivity shocks on unemployment. I leave this as a topic for further research.

7 Appendix

A.1 Properties of aggregate production function

Let $f(k) = \int_1^\infty xkg(x)e^{-k(1-G(x))}dx$. We show that $f'(k) > 0$, $f''(k) < 0$, $\lim_{k\to\infty} f'(k) = 0$ and $\lim_{k\to 0} f'(k) = E_G(x)$. Applying Leibniz's rule, we have

$$\begin{aligned} f'(k) &= \int_1^\infty \frac{\partial}{\partial k}\left(xkg(x)e^{-k(1-G(x))}\right)dx \qquad (14) \\ &= \int_1^\infty xg(x)(e^{-k(1-G(x))} - k(1-G(x))e^{-k(1-G(x))})dx \\ &= \int_1^\infty xg(x)e^{-k(1-G(x))}(1-k(1-G(x)))dx \\ &= \int_1^\infty xg(x)e^{-k(1-G(x))}dx - \int_1^\infty xg(x)e^{-k(1-G(x))}k(1-G(x))dx \end{aligned}$$

Rearranging and using integration by parts on the right integral, where $h_1(x) = x(1 - G(x))$ and $h_2(x) = e^{-k(1-G(x))}$, we have

$$\begin{aligned} &\int_1^\infty xg(x)e^{-k(1-G(x))}k(1-G(x))dx \\ =\ & \int_1^\infty h_1(x)h_2'(x)dx = [h_1(x)h_2(x)]_1^\infty - \int_1^\infty h_2(x)h_1'(x)dx \\ =\ & \left[x(1-G(x))e^{-k(1-G(x))}\right]_1^\infty - \int_1^\infty e^{-k(1-G(x))}((1-G(x)-xg(x))dx \\ =\ & -\left(e^{-k} + \int_1^\infty e^{-k(1-G(x))}((1-G(x)-xg(x))dx\right) \end{aligned}$$

where the last equality holds if we assume $\lim_{x\to\infty} x(1-G(x)) = 0$, since $\lim_{x\to\infty} e^{-k(1-G(x))} = 1$ and hence $\lim_{x\to\infty}\left(x(1-G(x))e^{-k(1-G(x))}\right) = 0$.

So we have

$$\begin{aligned} f'(k) &= \int_1^\infty xg(x)e^{-k(1-G(x))}dx - \int_1^\infty xg(x)e^{-k(1-G(x))}k(1-G(x))dx \\ &= \int_1^\infty xg(x)e^{-k(1-G(x))}dx + e^{-k} + \int_1^\infty e^{-k(1-G(x))}((1-G(x)-xg(x))dx \\ &= \int_1^\infty e^{-k(1-G(x))}(xg(x)\ + 1 - G(x) - xg(x))dx + e^{-k} \\ &= \int_1^\infty e^{-k(1-G(x))}(1-G(x))dx + e^{-k} > 0 \end{aligned}$$

Next we use Leibniz' rule to show that $f''(k) < 0$.

$$\begin{aligned} f''(k) &= \frac{d}{dk}\left(\int_1^\infty e^{-k(1-G(x))}(1-G(x))dx + e^{-k}\right) \\ &= \int_1^\infty \frac{\partial}{\partial k}\left(e^{-k(1-G(x))}(1-G(x))\right)dx - e^{-k} \\ &= -\left(\int_1^\infty (1-G(x))^2 e^{-k(1-G(x))}dx + e^{-k}\right) < 0 \end{aligned}$$

So we have shown that $f'(k) > 0$ and $f''(k) < 0$. It is clear that $\lim_{k\to\infty} f'(k) = 0$, since $f'(k) = \int_1^\infty e^{-k(1-G(x))}(1-G(x))dx + e^{-k}$.

We can also show that $\lim_{k\to 0} f'(k) = E_G(x) > 1$. From (14), we have $f'(k) = \int_1^\infty xg(x)e^{-k(1-G(x))}(1-k(1-G(x)))dx$. So we have

$$\begin{aligned} \lim_{k\to 0} f'(k) &= \lim_{k\to 0}\int_1^\infty xg(x)e^{-k(1-G(x))}(1-k(1-G(x)))dx \\ &= \int_1^\infty xg(x)dx = E_G(x) > 1. \end{aligned}$$

A.2 Production function for Pareto distribution

We show that $f(k) = k^\lambda \gamma(1-\lambda, k)$ if $G(x) = 1 - x^{-1/\lambda}$.

$$f(k) = \int_1^\infty xkg(x)e^{-k(1-G(x))}dx$$

Changing variables, letting $G(x) = y$, so that $x = (1-y)^{-\lambda}$, we have

$$f(k) = k\int_0^1 (1-y)^{-\lambda}e^{-k(1-y)}dy = k^{\lambda+1}\int_0^1 e^{-k(1-y)}(k(1-y))^{-\lambda}dz$$

Changing variables again, let $t = k(1-y)$, so $dy = -dt/k$

$$f(k) = -k^\lambda \int_k^0 t^{-\lambda}e^{-t}dt = k^\lambda \int_0^k t^{-\lambda}e^{-t}dt$$

Now, by definition, $\gamma(s,x) = \int_0^x t^{s-1}e^{-t}\, dt$, the Lower Incomplete Gamma function. Setting $x = k$, $s = 1-\lambda$, and using $k = \theta$, we have (3).

$$f(k) = \gamma(1-\lambda, \theta)k^\lambda$$

A.3 Derivation of elasticity of substitution

Starting with the definition of σ (in the form found in Arrow et al. (1961)) p. 229), the elasticity of substitution between capital and labor is as follows.

$$\sigma = \frac{-f'(k)(f(k) - kf'(k))}{kf(k)f''(k)}$$

Using the fact that $\frac{d}{dx}\gamma(s,x) = x^{s-1}e^{-x}$, we have $\frac{d}{dk}\gamma(1-\lambda,k) = k^{-\lambda}e^{-k}$ so $f'(k) = \lambda k^{\lambda-1}\gamma(1-\lambda,k) + e^{-k}$. To obtain $f''(k)$, we use the recurrence relation, $\gamma(s,x) = (s-1)\gamma(s-1,x) - x^{s-1}e^{-x}$, where $s = 2-\lambda$.

$$\begin{aligned}
f''(k) &= \frac{d}{dk}\left(\lambda k^{\lambda-1}\gamma(1-\lambda,k) + e^{-k}\right) \\
&= \lambda(\lambda-1)k^{\lambda-2}\gamma(1-\lambda,k) + \lambda k^{\lambda-1}k^{-\lambda}e^{-k} - e^{-k} \\
&= \lambda(\lambda-1)k^{\lambda-2}\gamma(1-\lambda,k) + \lambda k^{-1}e^{-k} - e^{-k} \\
&= -\lambda k^{\lambda-2}\left(\left(1-\lambda\right)\gamma(1-\lambda,k) - k^{1-\lambda}e^{-k}\right) - e^{-k} \\
&= -\left(\lambda k^{\lambda-2}\gamma(2-\lambda,k) + e^{-k}\right)
\end{aligned}$$

Inserting $f''(k) = -(\lambda k^{\lambda-2}\gamma(2-\lambda,k) + e^{-k})$ and $f'(k) = \lambda k^{\lambda-1}\gamma(1-\lambda,k) + e^{-k}$ into the formula for σ, we have

$$\begin{aligned}
\sigma &= \frac{-f'(k)(f(k) - kf'(k))}{kf(k)f''(k)} \\
&= \frac{-(\lambda k^{\lambda-1}\gamma(1-\lambda,k) + e^{-k})k^{\lambda}\gamma(2-\lambda,k)}{k^{\lambda+1}\gamma(1-\lambda,k)(-(\lambda k^{\lambda-2}\gamma(2-\lambda,k) + e^{-k}))}
\end{aligned}$$

Rearranging and simplifying, we have

$$\begin{aligned}
\sigma &= \frac{(\lambda k^{\lambda-1}\gamma(1-\lambda,k) + e^{-k})\gamma(2-\lambda,k)}{(\lambda k^{\lambda-1}\gamma(2-\lambda,k) + ke^{-k})\gamma(1-\lambda,k)} \\
&= \frac{\lambda k^{\lambda-1} + e^{-k}/\gamma(1-\lambda,k)}{\lambda k^{\lambda-1} + ke^{-k}/\gamma(2-\lambda,k)} \\
&= \frac{\lambda + k^{1-\lambda}e^{-k}/\gamma(1-\lambda,k)}{\lambda + k^{2-\lambda}e^{-k}/\gamma(2-\lambda,k)}
\end{aligned}$$

Now let $\varepsilon(s,x)$ be the elasticity of $\gamma(s,x)$ with respect to x, which is given by

$$\varepsilon(s,x) = \frac{d\gamma(s,x)}{dx}\frac{x}{\gamma(s,x)} = \frac{x^s e^{-x}}{\gamma(s,x)}$$

Using $k = \theta$, we have (6),

$$\sigma = \frac{\lambda + \varepsilon(1-\lambda,\theta)}{\lambda + \varepsilon(2-\lambda,\theta)}$$

A.4 Proof that elasticity of substitution $\sigma < 1$

To show that $\sigma < 1$, it is sufficient to show that $\varepsilon(1-\lambda,\theta) < \varepsilon(2-\lambda,\theta)$. To do this, we prove generally that $\varepsilon(s,x)$ is increasing in s for $x > 0$. First, by an application of Leibniz's integral rule, we have

$$\frac{\partial}{\partial s}\gamma(s,x) = \frac{\partial}{\partial s}\int_0^x t^{s-1}e^{-t}dt = \int_0^x \frac{\partial}{\partial s}\left(t^{s-1}e^{-t}\right)dt = -\int_0^x t^{s-1}e^{-t}(\ln t)dt \qquad (15)$$

Using this fact, we have

$$\begin{aligned}
\frac{\partial}{\partial s}\varepsilon(s,x) &= \frac{\partial}{\partial s}\left(\frac{x^s e^{-x}}{\gamma(s,x)}\right) \\
&= e^{-x}\left(\frac{x^s \ln x}{\gamma(s,x)} - \frac{x^s \int_0^x t^{s-1}e^{-t}(\ln t)dt}{\gamma(s,x)^2}\right) \\
&= x^s e^{-x}\left(\frac{\gamma(s,x)\ln x - \int_0^x t^{s-1}e^{-t}(\ln t)dt}{\gamma(s,x)^2}\right) \\
&= x^s e^{-x}\left(\frac{\ln x \int_0^x t^{s-1}e^{-t}dt - \int_0^x t^{s-1}e^{-t}(\ln t)dt}{\gamma(s,x)^2}\right) \\
&= x^s e^{-x}\left(\frac{\int_0^x (\ln x - \ln t)t^{s-1}e^{-t}dt}{\gamma(s,x)^2}\right)
\end{aligned}$$

Since $x > 0$ and $\ln x \geq \ln t$ for all $t \leq x$, we have $\frac{\partial}{\partial s}\varepsilon(s,x) > 0$ for any $x > 0$. Hence $\varepsilon(1-\lambda,\theta) < \varepsilon(2-\lambda,\theta)$ and therefore $\sigma < 1$.

A.5 Derivation of free-entry condition

Let n be the number of entrepreneurs arriving simultaneously at worker j. Let $\beta(x,n)$ be the probability of winning given a draw x from $G(x)$ and n entrepreneurs at worker j. Let $\pi(x,n)$ be the expected payoff for a successful entrepreneur with draw x and n entrepreneurs (bidders) approaching a given worker.

Let $R(x,n)$ equal the expected payoff (net of entry cost) for a bidder at worker j, given a draw x from $G(x)$ and n bidders. That is, $R(x,n) = \beta(x,n)\pi(x,n) - r$. From the perspective of the entrepreneur, we have $n \geq 1$ necessarily, so there are two cases to consider: *(i) bilateral match*: exactly one entrepreneur arrives at worker j.*(ii) multilateral match:* more than one entrepreneur arrives at worker j.

Case (i) $n = 1$. In this case, the wage paid to the worker is z. The expected payoff, given a draw x from $G(x)$, is therefore $R(x, 1) = x - z - r$. Integrating over the distribution $G(x)$, we have the expected payoff from a bilateral match, R_1.

$$R_1 = \int_1^{\infty} (x - z)\ dG(x) - r$$

Case (ii) $n \geq 2$. Suppose there is a fixed number $n \geq 2$ of entrepreneurs approaching a worker. In this case, $\pi(x, n) = (x - w(x, n))$, where $w(x, n)$ is the expected value of the second-best idea *given* that x is the best idea at worker j and there are n draws. We have

$$R(x, n) = \beta(x, n)(x - w(x, n)) - r \tag{16}$$

Here $w(x, n) = E(Y_2^n | Y_1^n = x)$, where Y_2^n is the second order statistic from n draws, and Y_1^n is the best idea from n draws. Let $H(y, n)$ be the distribution of Y_1^n, i.e. the distribution of the first-order statistic, which is just $H(y, n) = G(y)^n$.

Now $E(Y_2^n | Y_1^n = x) = E(Y_1^{n-1} | Y_1^{n-1} < x)$ where Y_1^{n-1} is the first order statistic for $n - 1$ draws (see p. 23 Krishna (2010)). Expected wages as a function of the best idea, x, and the number of bidders, n, can therefore be expressed as follows.

$$w(x, n) = \frac{1}{H(x,n-1)} \int_1^x y \quad dH(y, n - 1)$$

Substituting $w(x, n)$ into (16), we have

$$R(x, n) = \beta(x, n) \left(x - \frac{1}{H(x,n-1)} \int_1^x y \quad dH(y, n - 1) \right) - r$$

Now $\beta(x, n)$ is given as follows. Given n entrepreneurs at worker j, the probability that a given one of these entrepreneurs, with productivity draw x, has the best idea is simply $G(x)^{n-1}$. This is just the probability that the other $n - 1$ entrepreneurs at worker j all have ideas less than x. By assumption, $G(x)$ has no mass points, so the probability that two entrepreneurs draw identical ideas is zero. So $\beta(x, n) = G(x)^{n-1} = H(x, n - 1)$.

Substituting into the above, we obtain

$$R(x,n) = x.H(x,n-1) - \int_1^x y \ \ dH(y,n-1) - r$$

Using integration by parts, we get

$$R(x,n) = \int_1^x H(y,n-1)dy - r$$

The expected payoff from approaching worker j is given by

$$R(n) = \int_1^\infty R(x,n)g(x)dx - r$$

Again integrating by parts, we obtain

$$R(n) = [R(x,n)G(x)]_1^\infty - \int_1^\infty \frac{d}{dx}[R(x,n)]G(x)dx$$

Now, $\frac{d}{dx}[R(x,n)]$ can be determined as follows.

$$\begin{aligned}
\frac{d}{dx}[R(x,n)] &= \frac{d}{dx}\left(\int_1^x H(y,n-1)dy - r\right) \\
&= H(x,n-1) - H(1,n-1) \\
&= H(x,n-1) - G(1)^{n-1} = H(x,n-1)
\end{aligned}$$

Also, $[R(x,n)G(x)]_1^\infty = \lim_{x\to\infty} R(x,n) = \int_1^\infty H(y,n-1)dy - r$, since $G(x) \to 1$ as $x \to \infty$ and $G(1) = 0$. So we have

$$R(n) = \int_1^\infty H(y,n-1)dy - \int_1^\infty H(x,n-1)G(x)dx - r$$

Rearranging and substituting $H(x,n-1) = \beta(x,n)$, the following holds for all $n \geq 2$.

$$R(n) = \int_1^\infty \beta(x,n)(1-G(x))dx - r \tag{17}$$

We can now determine $R_2(\theta)$, the expected payoff from a multilateral match, i.e. the expected payoff given that $n \geq 2$.

$$
\begin{aligned}
R_2(\theta) &= \frac{1}{\Pr(n_j \geq 2)} \sum_{n=2}^{\infty} \Pr(n_j = n) R(n) \\
&= \frac{1}{\Pr(n_j \geq 2)} \sum_{n=2}^{\infty} \Pr(n_j = n) \int_1^{\infty} \beta(x,n)(1 - G(x))dx - r \\
&= \int_1^{\infty} \frac{1}{\Pr(n_j \geq 2)} \sum_{n=2}^{\infty} \Pr(n_j = n)\beta(x,n)(1 - G(x))dx - r \\
&= \int_1^{\infty} \beta(x)(1 - G(x))dx - r
\end{aligned}
$$

where $\beta(x)$ is the probability of winning given that $n \geq 2$.

The probability $\beta(x)$ be determined as follows. Let $\Pr(n_j = n)$ be the probability of n arrivals from the perspective of the entrepreneurs.

$$
\beta(x) = \frac{1}{\Pr(n_j \geq 2)} \sum_{n=2}^{\infty} \Pr(n_j = n)\beta(x,n) = \frac{1}{\Pr(n_j \geq 2)} \sum_{n=2}^{\infty} \Pr(n_j = n) G(x)^{n-1}
$$

From the perspective of *workers*, the probability of n arrivals is given by the Poisson distribution, namely $\Pr(\hat{n}_j = n) = e^{-\theta}\theta^n/n!$ From the perspective of *entrepreneurs*, however, this must be weighted appropriately. The probability of n arrivals from the perspective of the entrepreneur is given by

$$
\Pr(n_j = n) = \Pr(\hat{n}_j = n)\frac{n}{E(\hat{n}_j)} = \frac{e^{-\theta}\theta^n}{n!}\frac{n}{\theta} = \frac{e^{-\theta}\theta^{n-1}}{(n-1)!}
$$

From the entrepreneurs' perspective, $\Pr(n_j \geq 2) = 1 - \Pr(n_j = 1) = 1 - e^{-\theta}$. So the probability of winning, $\beta(x)$, is given as follows.

$$
\begin{aligned}
\beta(x) &= \frac{1}{1 - e^{-\theta}} \sum_{n=2}^{\infty} \frac{e^{-\theta}\theta^{n-1}}{(n-1)!} G(x)^{n-1} \\
&= \frac{e^{-\theta}}{1 - e^{-\theta}} \sum_{n=1}^{\infty} \frac{(\theta G(x))^n}{n!} \\
&= \frac{e^{-\theta}}{1 - e^{-\theta}} \left(\sum_{n=0}^{\infty} \frac{(\theta G(x))^n}{n!} - 1 \right) \\
&= \frac{e^{-\theta(1-G(x))} - e^{-\theta}}{1 - e^{-\theta}}
\end{aligned}
$$

Substituting $\beta(x)$ into the expression for $R_2(\theta)$ above, we get

$$R_2(\theta) = \int_1^\infty \frac{e^{-\theta(1-G(x))} - e^{-\theta}}{1-e^{-\theta}}(1-G(x))dx - r$$

Now let $R(\theta)$ be the expected payoff for an entrepreneur as a function of θ alone. Again considering the probability of n entrepreneurs arriving from the perspective of the entrepreneurs, we have $R(\theta) = \Pr(n_j \geq 2)R_2(\theta) + \Pr(n_j = 1)R_1$, so

$$R(\theta) = (1-e^{-\theta})\int_1^\infty \left(\frac{e^{-\theta(1-G(x))} - e^{-\theta}}{1-e^{-\theta}}\right)(1-G(x))dx + e^{-\theta}\int_1^\infty (x-z)\, dG(x) - r$$

Rearranging, we get

$$R(\theta) = \int_1^\infty e^{-\theta(1-G(x))}(1-G(x))dx + e^{-\theta}\left(\left(\int_1^\infty x\, g(x)dx - \int_1^\infty (1-G(x))dx\right) - z\right) - r$$

Now, using integration by parts, where $h_1(x) = x$ and $h_2(x) = -(1-G(x))$,we have

$$\begin{aligned} & \int_1^\infty x\, g(x)dx - \int_1^\infty (1-G(x))dx \\ = & \int_1^\infty h_1(x)h_2'(x)dx + \int_1^\infty h_1'(x)h_2(x)dx = [h_1(x)h_2(x)]_1^\infty \\ = & -[x(1-G(x))]_1^\infty = 1 \end{aligned}$$

where the last equality holds if we assume that $\lim_{x\to\infty} x(1-G(x)) = 0$. Using the same notation as the main text, $R(\theta) = \pi(\theta)$ and we have equation (7).

$$\pi(\theta) = \int_1^\infty e^{-\theta(1-G(x))}(1-G(x))dx + e^{-\theta}(1-z) - r$$

A.6 Proof of Proposition 1

Existence. Let $F(\theta) = \int_1^\infty e^{-\theta(1-G(x))}(1-G(x))dx + e^{-\theta}(1-z)$. Clearly, the free entry condition, $\pi(\theta) = 0$, holds if and only if $F(\theta) = r$, where $r > 0$. Now $F(\theta)$ is continuous in θ on $[0, \infty)$ and $F(\theta) \to 0$ as $\theta \to \infty$. If we can ensure that $F(0) > r$, the intermediate value theorem implies there must exist a $\theta > 0$ such that $F(\theta) = r$. Now, $F(0) = \int_1^\infty (1-G(x))dx + (1-z) = E_G(x) - 1 + 1 - z = E_G(x) - z$. So we can ensure that $F(0) > r$ provided we assume $E_G(x) > z + r$, which is Assumption 1.

Uniqueness. To prove uniqueness of the equilibrium θ^* such that $F(\theta) = r$, it suffices to show that $F'(\theta) < 0$. Now let $f(\theta, x) = e^{-\theta(1-G(x))}(1 - G(x))$. Since both $f(\theta, x)$ and $\frac{\partial}{\partial\theta}(f(\theta, x))$ are continuous in both θ and x on $[1, \infty)$, we can use Leibniz' integral rule, which implies that $F'(\theta) = \int_1^\infty \frac{\partial}{\partial\theta}(f(\theta, x))dx - (1 - z)e^{-\theta} = -\int_1^\infty (1 - G(x))^2 e^{-\theta(1-G(x))}dx - (1 - z)e^{-\theta}$. Clearly, $F'(\theta) < 0$, so there exists a unique θ^* such that $F(\theta) = r$.

A.7 Proof of comparative statics for θ^* for any $G(x)$

Proof that $\theta'(z) < 0$. Let $F(z, \theta) = \int_1^\infty e^{-\theta(1-G(x))}(1 - G(x))dx + e^{-\theta}(1 - z) - r = 0$. By the implicit function theorem, $\theta'(z) = -\frac{\partial F/\partial z}{\partial F/\partial\theta}$. Now $\frac{\partial F}{\partial z} = -e^{-\theta}$ and

$$\frac{\partial F}{\partial\theta} = -\int_1^\infty e^{-\theta(1-G(x))}(1 - G(x))^2 dx - e^{-\theta}(1 - z)$$

Now since $z \leq 1$, we have

$$\theta'(z) = -\frac{\partial F/\partial z}{\partial F/\partial\theta} = \frac{-e^{-\theta}}{\left(\int_1^\infty e^{-\theta(1-G(x))}(1 - G(x))^2 dx + e^{-\theta}(1 - z)\right)} < 0$$

Proof that $\theta'(r) < 0$. Let $F(z, \theta) = \int_1^\infty e^{-\theta(1-G(x))}(1-G(x))dx + e^{-\theta}(1 - z) - r = 0$. By the implicit function theorem, $\theta'(r) = -\frac{\partial F/\partial r}{\partial F/\partial\theta}$ where $\frac{\partial F}{\partial r} = -1$. From above, we have $\frac{\partial F}{\partial\theta} = -\int_1^\infty e^{-\theta(1-G(x))}(1 - G(x))^2 dx - e^{-\theta}(1 - z)$, so

$$\theta'(r) = -\frac{\partial F/\partial r}{\partial F/\partial\theta} = \frac{-1}{\left(\int_1^\infty e^{-\theta(1-G(x))}(1 - G(x))^2 dx + e^{-\theta}(1 - z)\right)} < 0$$

A.8 Free entry condition – Pareto distribution

Starting with the free entry condition and $G(x) = 1 - x^{-1/\lambda}$, let $y = G(x)$ to obtain:

$$\begin{aligned}\pi(\theta) &= \lambda\int_0^1 e^{-\theta(1-y)}(1 - y)^{-\lambda}dy + e^{-\theta}(1 - z) - r \\ &= \lambda\theta^\lambda \int_0^1 e^{-\theta(1-y)}(\theta(1 - y))^{-\lambda}dy + e^{-\theta}(1 - z) - r\end{aligned}$$

Changing variables, let $t = \theta(1-y)$, so $dy = -dt/\theta$

$$\pi(\theta) = -\lambda\theta^{\lambda-1}\int_\theta^0 t^{-\lambda}e^{-t}dt - e^{-\theta}(1-z) - r = \lambda\theta^{\lambda-1}\int_0^\theta t^{-\lambda}e^{-t}dt + e^{-\theta}(1-z) - r$$

Now, by definition $\gamma(s,x) = \int_0^x t^{s-1}e^{-t}\ dt$, the Lower Incomplete Gamma function. Setting $x = \theta$ and $s = 1-\lambda$, we have (9).

$$\pi(\theta) = \lambda\theta^{\lambda-1}\gamma(1-\lambda,\theta) + e^{-\theta}(1-z) - r = 0$$

A.9 Proof of comparative statics for θ^* – Pareto distribution

We have already shown in Appendix A.7 that $\theta'(z) < 0$ and $\theta'(r) < 0$ for *any* distribution $G(x)$. However, we include the derivations for the Pareto distribution here because the expressions for $\theta'(z)$ and $\theta'(r)$ will be used in subsequent proofs.

Proof that $\theta'(z) < 0$. Let $F(z,\theta) = \lambda\theta^{\lambda-1}\gamma(1-\lambda,\theta) + (1-z)e^{-\theta} - r = 0$. By the implicit function theorem, $\theta'(z) = -\frac{\partial F/\partial z}{\partial F/\partial \theta}$. Now $\frac{\partial F}{\partial z} = -e^{-\theta}$ and

$$\begin{aligned}
\frac{\partial F}{\partial \theta} &= \lambda(\lambda-1)\theta^{\lambda-2}\gamma(1-\lambda,\theta) + \lambda\theta^{\lambda-1}\theta^{-\lambda}e^{-\theta} - (1-z)e^{-\theta} \\
&= -\lambda\theta^{\lambda-2}((1-\lambda)\gamma(1-\lambda,\theta) - \theta^{1-\lambda}e^{-\theta}) - (1-z)e^{-\theta} \\
&= -\lambda\theta^{\lambda-2}\gamma(2-\lambda,\theta) - (1-z)e^{-\theta}
\end{aligned}$$

Here we use the recurrence relation, $\gamma(2-\lambda,\theta) = (1-\lambda)\gamma(1-\lambda,\theta) - \theta^{1-\lambda}e^{-\theta}$. Now since $z \leq 1$, we have

$$\theta'(z) = -\frac{\partial F/\partial z}{\partial F/\partial \theta} = \frac{-e^{-\theta}}{\lambda\theta^{\lambda-2}\gamma(2-\lambda,\theta) + (1-z)e^{-\theta}} < 0$$

Proof that $\theta'(r) < 0$. Let $F(z,\theta) = \lambda\theta^{\lambda-1}\gamma(1-\lambda,\theta) + (1-z)e^{-\theta} - r = 0$. By the implicit function theorem, $\theta'(r) = -\frac{\partial F/\partial r}{\partial F/\partial \theta}$ where $\frac{\partial F}{\partial r} = -1$. From above, $\frac{\partial F}{\partial \theta}$ $= -\lambda\theta^{\lambda-2}\gamma(2-\lambda,\theta) - (1-z)e^{-\theta}$. So we have

$$\theta'(r) = -\frac{\partial F/\partial r}{\partial F/\partial \theta} = \frac{-1}{\lambda\theta^{\lambda-2}\gamma(2-\lambda,\theta) + (1-z)e^{-\theta}} < 0$$

Proof that $\theta'(\lambda) > 0$. Let $F(\lambda,\theta) = \lambda\theta^{\lambda-1}\gamma(1-\lambda,\theta) + (1-z)e^{-\theta} - r = 0$. By the implicit function theorem, $\frac{d\theta^*}{d\lambda} = -\frac{\partial F/\partial\lambda}{\partial F/\partial\theta}$. Since $\frac{\partial F}{\partial \theta} < 0$, in order to show that $\frac{d\theta^*}{d\lambda} > 0$ it suffices to show that $\frac{\partial F}{\partial \lambda} > 0$.

$$\begin{aligned}
\frac{\partial F}{\partial\lambda} &= \frac{\partial}{\partial\lambda}\left(\lambda\theta^{\lambda-1}\gamma(1-\lambda,\theta)\right) \\
&= \theta^{\lambda-1}\gamma(1-\lambda,\theta) + \lambda\left(\theta^{\lambda-1}(\ln\theta)\gamma(1-\lambda,\theta) + \theta^{\lambda-1}\frac{\partial}{\partial\lambda}\gamma(1-\lambda,\theta)\right)
\end{aligned}$$

Using (15),

$$\frac{\partial}{\partial\lambda}\gamma(1-\lambda,\theta) = -\int_0^\theta t^{-\lambda}e^{-t}(\ln t)dt$$

So we have

$$\begin{aligned}
\frac{\partial F}{\partial\lambda} &= \theta^{\lambda-1}\gamma(1-\lambda,\theta) + \lambda\left(\theta^{\lambda-1}(\ln\theta)\gamma(1-\lambda,\theta) - \theta^{\lambda-1}\int_0^\theta t^{-\lambda}e^{-t}(\ln t)dt\right) \\
&= \theta^{\lambda-1}\gamma(1-\lambda,\theta) + \lambda\theta^{\lambda-1}\left(\ln\theta\int_0^\theta t^{-\lambda}e^{-t}\,dt - \int_0^\theta t^{-\lambda}e^{-t}(\ln t)dt\right) \\
&= \theta^{\lambda-1}\gamma(1-\lambda,\theta) + \lambda\theta^{\lambda-1}\int_0^\theta t^{-\lambda}e^{-t}(\ln\theta - \ln t)dt
\end{aligned}$$

Since $\theta \geq t$ for all $t \in [0,\theta]$, $\ln\theta - \ln t \geq 0$ for all $t \in [0,\theta]$, so $t^{-\lambda}e^{-t}(\ln\theta - \ln t) \geq 0$ for all $t \in [0,\theta]$ and hence $\int_0^\theta t^{-\lambda}e^{-t}(\ln\theta - \ln t)dt \geq 0$. So we have $\frac{d\theta^*}{d\lambda} > 0$. From above, $\frac{\partial F}{\partial\theta} = -\lambda\theta^{\lambda-2}\gamma(2-\lambda,\theta) - (1-z)e^{-\theta}$. So we have

$$\frac{d\theta^*}{d\lambda} = -\frac{\partial F/\partial\lambda}{\partial F/\partial\theta} = \frac{\theta^{\lambda-1}\left(\gamma(1-\lambda,\theta) + \lambda\int_0^\theta t^{-\lambda}e^{-t}(\ln\theta - \ln t)dt\right)}{\lambda\theta^{\lambda-2}\gamma(2-\lambda,\theta) + (1-z)e^{-\theta}} \tag{18}$$

Proof that $\frac{df}{d\lambda} > 0$. Let $f(\theta(\lambda),\lambda) = \theta^\lambda\gamma(1-\lambda,\theta)$. Then $\frac{df}{d\lambda} = \frac{\partial f}{\partial\theta}\theta'(\lambda) + \frac{\partial f}{\partial\lambda}$. Now $f'(\theta) > 0$ and $\theta'(\lambda) > 0$ from above, so it suffices to show that $\frac{\partial f}{\partial\lambda} > 0$.

$$\begin{aligned}
\frac{\partial f}{\partial\lambda} &= \frac{\partial}{\partial\lambda}\left(\theta^\lambda\gamma(1-\lambda,\theta)\right) \\
&= \theta^\lambda\ln\theta\gamma(1-\lambda,\theta) + \theta^\lambda\frac{\partial}{\partial\lambda}\gamma(1-\lambda,\theta) \\
&= \theta^\lambda\ln\theta\gamma(1-\lambda,\theta) - \theta^\lambda\int_0^\theta t^{-\lambda}e^{-t}(\ln t)dt \\
&= \theta^\lambda\left(\int_0^\theta t^{-\lambda}e^{-t}(\ln\theta - \ln t)dt\right) > 0
\end{aligned}$$

A.10 Proof of constrained efficiency result

The first order condition for the social planner's problem requires that $f'(\theta_P) = r + ze^{-\theta}$, while the free entry condition for the decentralized equilibrium requires that $\displaystyle\int_1^\infty e^{-\theta(1-G(x))}(1-G(x))dx + e^{-\theta} = r + ze^{-\theta}$. In Appendix A.1, we showed that $f'(\theta) = \int_1^\infty e^{-\theta(1-G(x))}(1-G(x))dx + e^{-\theta}$, so clearly we have $\theta_P = \theta^*$ and constrained efficiency holds.

A.11 Proof that capital share, s_K, is decreasing in θ

Since $s_K = \lambda + (1-z)\varepsilon(1-\lambda, \theta)$, it is sufficient to prove that $\varepsilon(1-\lambda,\theta)$ is decreasing in θ. Note that here we are considering θ *outside* of equilibrium.

Let $h(\theta) = \varepsilon(1-\lambda,\theta) = \frac{\theta^{1-\lambda}e^{-\theta}}{\gamma(1-\lambda,\theta)}$. Differentiating $h(\theta)$, we have

$$\begin{aligned} h'(\theta) &= ((1-\lambda)\theta^{-\lambda}e^{-\theta} - \theta^{1-\lambda}e^{-\theta})\gamma(1-\lambda,\theta)^{-1} - \theta^{1-\lambda}e^{-\theta}\gamma(1-\lambda,\theta)^{-2}\theta^{-\lambda}e^{-\theta} \quad (19)\\ &= ((1-\lambda)\theta^{-\lambda}e^{-\theta} - \theta^{1-\lambda}e^{-\theta})\gamma(1-\lambda,\theta)^{-1} - \theta^{1-2\lambda}e^{-2\theta}\gamma(1-\lambda,\theta)^{-2} \\ &= \frac{\theta^{-\lambda}e^{-\theta}}{\gamma(1-\lambda,\theta)}\left(1-\lambda-\theta-\frac{\theta^{1-\lambda}e^{-\theta}}{\gamma(1-\lambda,\theta)}\right) \end{aligned}$$

Hence $h'(\theta) < 0$ if and only if $\theta > 1-\lambda-\frac{\theta^{1-\lambda}e^{-\theta}}{\gamma(1-\lambda,\theta)}$. Using the recurrence relation, $\gamma(2-\lambda,\theta) = (1-\lambda)\gamma(1-\lambda,\theta) - \theta^{1-\lambda}e^{-\theta}$, we have $h'(\theta) < 0$ if and only if

$$\begin{aligned} \theta &> \frac{\gamma(2-\lambda,\theta)}{\gamma(1-\lambda,\theta)} \\ &\Leftrightarrow \frac{\theta^{2-\lambda}e^{-\theta}}{\gamma(2-\lambda,\theta)} > \frac{\theta^{1-\lambda}e^{-\theta}}{\gamma(1-\lambda,\theta)} \\ &\Leftrightarrow \varepsilon(1-\lambda,\theta) < \varepsilon(2-\lambda,\theta) \end{aligned}$$

So we have $h'(\theta) < 0$ if and only if $\varepsilon(1-\lambda,\theta) < \varepsilon(2-\lambda,\theta)$. From Appendix A.4, we have $\varepsilon(s,x)$ is increasing in s, so $\varepsilon(1-\lambda,\theta) < \varepsilon(2-\lambda,\theta)$. Hence we conclude that $h'(\theta) < 0$.

To establish upper and lower bounds, recall that as $\theta \to 0$, we have $\varepsilon(s,x) \to s$, and as $\theta \to \infty$, we have $\varepsilon(1-\lambda,\theta) \to 0$. This means that as $\theta \to 0$, we have $s_K = 1 - z(1-\lambda)$, and as $\theta \to \infty$, we have $s_K = \lambda$.

A.12 Proof of Proposition 3

We show that $\frac{ds_K^*}{dz} < 0$ where $s_K^* = \lambda + \frac{(1-z)\theta^{1-\lambda}e^{-\theta}}{\gamma(1-\lambda,\theta)}$ where θ^* solves $\lambda\theta^{\lambda-1}\gamma(1-\lambda,\theta) + (1-z)e^{-\theta} = r$. So $s_K^* = \lambda + (1-z)h(\theta(z))$ where $h(\theta) = \varepsilon(1-\lambda,\theta) = \frac{\theta^{1-\lambda}e^{-\theta}}{\gamma(1-\lambda,\theta)}$, and

$$\begin{aligned}\frac{ds_K^*}{dz} &= \frac{\partial}{\partial z}(\lambda + (1-z)h(\theta)) + \theta'(z)\frac{\partial}{\partial\theta}(\lambda + (1-z)h(\theta)) \\ &= -h(\theta) + (1-z)\theta'(z)h'(\theta)\end{aligned}$$

Now, from (19), $h'(\theta)$ is the following

$$h'(\theta) = \frac{\theta^{-\lambda}e^{-\theta}}{\gamma(1-\lambda,\theta)}\left(1-\lambda-\theta-\frac{\theta^{1-\lambda}e^{-\theta}}{\gamma(1-\lambda,\theta)}\right)$$

So we have

$$\begin{aligned}\frac{ds_K^*}{dz} &= \frac{-\theta^{1-\lambda}e^{-\theta}}{\gamma(1-\lambda,\theta)} + (1-z)\theta'(z)\frac{\theta^{-\lambda}e^{-\theta}}{\gamma(1-\lambda,\theta)}\left(1-\lambda-\theta-\frac{\theta^{1-\lambda}e^{-\theta}}{\gamma(1-\lambda,\theta)}\right) \\ &= \frac{\theta^{-\lambda}e^{-\theta}}{\gamma(1-\lambda,\theta)}\left(-\theta + (1-z)\theta'(z)\left(1-\lambda-\theta-\frac{\theta^{1-\lambda}e^{-\theta}}{\gamma(1-\lambda,\theta)}\right)\right)\end{aligned}$$

This means that $\frac{ds_K^*}{dz} < 0$ if and only if

$$\theta > (1-z)\theta'(z)\left(1-\lambda-\theta-\frac{\theta^{1-\lambda}e^{-\theta}}{\gamma(1-\lambda,\theta)}\right) \tag{20}$$

Now we have

$$\begin{aligned}&(1-z)\theta'(z)\left(1-\lambda-\theta-\frac{\theta^{1-\lambda}e^{-\theta}}{\gamma(1-\lambda,\theta)}\right) \\ =\ &\theta'(z)\left((1-\lambda)(1-z) - \theta(1-z) - \frac{(1-z)\theta^{1-\lambda}e^{-\theta}}{\gamma(1-\lambda,\theta)}\right) \\ =\ &\theta'(z)\left(1-\lambda-\frac{(1-z)\theta^{1-\lambda}e^{-\theta}}{\gamma(1-\lambda,\theta)} - z(1-\lambda) - \theta(1-z)\right) \\ =\ &\theta'(z)\left(s_L - z(1-\lambda) - \theta(1-z)\right)\end{aligned}$$

Since $\theta'(z) < 0$, the above inequality (20) is equivalent to

$$\frac{\theta}{\theta'(z)} < (s_L - z(1-\lambda) - \theta(1-z))$$

Now $s_L \geq z(1-\lambda)$, the lower bound on labor's share, so it is sufficient to show that $-\theta(1-z) > \frac{\theta}{\theta'(z)}$. Since $\theta'(z) < 0$, this holds if and only if $\theta'(z) > \frac{-1}{1-z}$.

$$\begin{aligned} \theta'(z) &= \frac{-e^{-\theta}}{\lambda\theta^{\lambda-2}\gamma(2-\lambda,\theta)+(1-z)e^{-\theta}} > \frac{-1}{1-z} \\ &\Leftrightarrow \frac{1}{e^{\theta}\lambda\theta^{\lambda-2}\gamma(2-\lambda,\theta)+(1-z)} < \frac{1}{1-z} \end{aligned}$$

This holds exactly when $e^{\theta}\lambda\theta^{\lambda-2}\gamma(2-\lambda,\theta) > 0$, which is true. So we have $\frac{ds_K^*}{dz} < 0$.

A.13 Proof of Proposition 4

Let $w(\theta^*) = (1-\lambda)\theta^{\lambda}\gamma(1-\lambda,\theta) - (1-z)\theta e^{-\theta}$, where θ^* solves $\lambda\theta^{\lambda-1}\gamma(1-\lambda,\theta) + (1-z)e^{-\theta} = r$. Now $w'(z) = \frac{\partial w}{\partial\theta}\theta'(z) + \frac{\partial w}{\partial z}$, where $\frac{\partial w}{\partial z} = \theta e^{-\theta}$ and

$$\begin{aligned} \frac{\partial w}{\partial \theta} &= (1-\lambda)\lambda\theta^{\lambda-1}\gamma(1-\lambda,\theta) + (1-\lambda)\theta^{\lambda}\theta^{-\lambda}e^{-\theta} - (1-z)(e^{-\theta} - \theta e^{-\theta}) \\ &= (1-\lambda)\lambda\theta^{\lambda-1}\gamma(1-\lambda,\theta) + (1-\lambda)e^{-\theta} - (1-z)(e^{-\theta} - \theta e^{-\theta}) \\ &= (1-\lambda)\left(\lambda\theta^{\lambda-1}\gamma(1-\lambda,\theta) + e^{-\theta}\right) - (1-z)e^{-\theta}(1-\theta) \end{aligned}$$

So we have

$$w'(z) = \frac{\partial w}{\partial \theta}\theta'(z) + \frac{\partial w}{\partial z} = \left((1-\lambda)\left(\lambda\theta^{\lambda-1}\gamma(1-\lambda,\theta) + e^{-\theta}\right) - (1-z)e^{-\theta}(1-\theta)\right)\theta'(z) + \theta e^{-\theta}$$

Now $w'(z) \leq 0$ if and only if

$$\begin{aligned} \left((1-\lambda)\left(\lambda\theta^{\lambda-1}\gamma(1-\lambda,\theta) + e^{-\theta}\right) - (1-z)e^{-\theta}(1-\theta)\right)\theta'(z) + \theta e^{-\theta} &\leq 0 \Leftrightarrow \\ \left((1-\lambda)\left(\lambda\theta^{\lambda-1}\gamma(1-\lambda,\theta) + e^{-\theta}\right) - (1-z)e^{-\theta}(1-\theta)\right) + \frac{\theta e^{-\theta}}{\theta'(z)} &\geq 0 \end{aligned}$$

since $\theta'(z) < 0$. So we have $w'(z) \leq 0$ if and only if

$$\frac{-\theta e^{-\theta}}{\theta'(z)} \leq (1-\lambda)\lambda\theta^{\lambda-1}\gamma(1-\lambda,\theta) + \left(1-\lambda+(1-z)(\theta-1)\right)e^{-\theta}$$

From above, we have

$$\begin{aligned}\theta'(z) &= \frac{-e^{-\theta}}{\lambda\theta^{\lambda-2}\gamma(2-\lambda,\theta)+(1-z)e^{-\theta}} \\ &= \frac{-e^{-\theta}}{\lambda\theta^{\lambda-2}((1-\lambda)\gamma(1-\lambda,\theta)-\theta^{1-\lambda}e^{-\theta})+(1-z)e^{-\theta}}\end{aligned}$$

So we have

$$\begin{aligned}\frac{-\theta e^{-\theta}}{\theta'(z)} &= \frac{-\theta e^{-\theta}\left(\lambda\theta^{\lambda-2}((1-\lambda)\gamma(1-\lambda,\theta)-\theta^{1-\lambda}e^{-\theta})+(1-z)e^{-\theta}\right)}{-e^{-\theta}} \\ &= \lambda\theta^{\lambda-1}((1-\lambda)\gamma(1-\lambda,\theta)-\theta^{1-\lambda}e^{-\theta})+\theta(1-z)e^{-\theta} \\ &= \lambda(1-\lambda)\theta^{\lambda-1}\gamma(1-\lambda,\theta)+(-\lambda+\theta(1-z))e^{-\theta}\end{aligned}$$

So $w'(z) \leq 0$ if and only if

$$-\lambda+\theta(1-z) \leq 1-\lambda+(1-z)(\theta-1)$$

Re-arranging, this condition holds if and only if $z \geq 0$, which is true.

A.14 Proof of Proposition 5

Let $s_K^* = \lambda + \frac{(1-z)\theta^{1-\lambda}e^{-\theta}}{\gamma(1-\lambda,\theta)}$ where $\theta^*(\lambda)$ solves $\lambda\theta^{\lambda-1}\gamma(1-\lambda,\theta)+(1-z)e^{-\theta} = r$. Rearranging the zero profit condition and substituting into the expression for capital share, we get

$$s_K^* = \frac{r\theta^{1-\lambda}}{\gamma(1-\lambda,\theta)}$$

So we have

$$\frac{ds_K^*}{d\lambda} = r\frac{\partial}{\partial\theta}\left(\frac{\theta^{1-\lambda}}{\gamma(1-\lambda,\theta)}\right)\theta'(\lambda) + r\frac{\partial}{\partial\lambda}\left(\frac{\theta^{1-\lambda}}{\gamma(1-\lambda,\theta)}\right)$$

Using the fact that $\frac{d}{d\theta}\gamma(1-\lambda,\theta) = \theta^{-\lambda}e^{-\theta}$, we have

$$\frac{\partial}{\partial\theta}\left(\frac{\theta^{1-\lambda}}{\gamma(1-\lambda,\theta)}\right) = \frac{(1-\lambda)\theta^{-\lambda}}{\gamma(1-\lambda,\theta)} - \frac{\theta^{1-2\lambda}e^{-\theta}}{\gamma(1-\lambda,\theta)^2}$$

Using (15),

$$\begin{aligned}
\frac{\partial}{\partial \lambda}\left(\frac{\theta^{1-\lambda}}{\gamma(1-\lambda,\theta)}\right) &= \frac{-\theta^{1-\lambda}\ln\theta}{\gamma(1-\lambda,\theta)} - \frac{\theta^{1-\lambda}\frac{d}{d\lambda}\gamma(1-\lambda,\theta)}{\gamma(1-\lambda,\theta)^2} \\
&= \frac{-\theta^{1-\lambda}\ln\theta}{\gamma(1-\lambda,\theta)} + \frac{\theta^{1-\lambda}\displaystyle\int_0^\theta t^{-\lambda}e^{-t}\ln t\; dt}{\gamma(1-\lambda,\theta)^2} \\
&= \frac{-\theta^{1-\lambda}}{\gamma(1-\lambda,\theta)^2}\left(\gamma(1-\lambda,\theta)\ln\theta - \int_0^\theta t^{-\lambda}e^{-t}\ln t\; dt\right) \\
&= \frac{-\theta^{1-\lambda}}{\gamma(1-\lambda,\theta)^2}\left(\int_0^\theta t^{-\lambda}e^{-t}(\ln\theta - \ln t)\; dt\right)
\end{aligned}$$

Letting $B = \int_0^\theta t^{-\lambda}e^{-t}(\ln\theta - \ln t)dt$, we have the following:

$$\begin{aligned}
\frac{ds_K^*}{d\lambda} &= r\frac{\partial}{\partial\theta}\left(\frac{\theta^{1-\lambda}}{\gamma(1-\lambda,\theta)}\right)\theta'(\lambda) + r\frac{\partial}{\partial\lambda}\left(\frac{\theta^{1-\lambda}}{\gamma(1-\lambda,\theta)}\right) \\
&= r\left(\frac{(1-\lambda)\theta^{-\lambda}}{\gamma(1-\lambda,\theta)} - \frac{\theta^{1-2\lambda}e^{-\theta}}{\gamma(1-\lambda,\theta)^2}\right)\theta'(\lambda) - r\frac{\theta^{1-\lambda}B}{\gamma(1-\lambda,\theta)^2} \\
&= \frac{r\theta^{-\lambda}}{\gamma(1-\lambda,\theta)}\left(\left((1-\lambda) - \frac{\theta^{1-\lambda}e^{-\theta}}{\gamma(1-\lambda,\theta)}\right)\theta'(\lambda) - \frac{\theta B}{\gamma(1-\lambda,\theta)}\right)
\end{aligned}$$

So $\frac{ds_K^*}{d\lambda} > 0$ if and only if the following holds:

$$\left((1-\lambda) - \frac{\theta^{1-\lambda}e^{-\theta}}{\gamma(1-\lambda,\theta)}\right)\theta'(\lambda) > \frac{\theta B}{\gamma(1-\lambda,\theta)}$$

Applying the recurrence relation $\gamma(s,x) = (s-1)\gamma(s-1,x) - x^{s-1}e^{-x}$, we have

$$(1-\lambda) - \frac{\theta^{1-\lambda}e^{-\theta}}{\gamma(1-\lambda,\theta)} = \frac{\gamma(2-\lambda,\theta)}{\gamma(1-\lambda,\theta)}$$

So we have $\frac{ds_K^*}{d\lambda} > 0$ if and only if $\gamma(2-\lambda,\theta)\theta'(\lambda) > \theta B$. Now, we have already shown above that

$$\frac{d\theta^*}{d\lambda} = \frac{\theta^{\lambda-1}\left(\gamma(1-\lambda,\theta) + \lambda B\right)}{\lambda\theta^{\lambda-2}\gamma(2-\lambda,\theta) + (1-z)e^{-\theta}}$$

So $\frac{ds_K^*}{d\lambda} > 0$ if and only if

$$\gamma(2-\lambda,\theta)\left(\frac{\theta^{\lambda-1}\left(\gamma(1-\lambda,\theta) + \lambda B\right)}{\lambda\theta^{\lambda-2}\gamma(2-\lambda,\theta) + (1-z)e^{-\theta}}\right) > \theta B$$

Re-arranging and simplifying, this is equivalent to

$$\gamma(2-\lambda,\theta)\gamma(1-\lambda,\theta) > B(1-z)\theta^{2-\lambda}e^{-\theta} \tag{21}$$

Calculating the integral in B, we have

$$B = \int_0^\theta t^{-\lambda}e^{-t}(\ln\theta - \ln t)dt = \frac{\theta^{1-\lambda}}{(1-\lambda)^2}F_{2,2}(1-\lambda, 1-\lambda; 2-\lambda, 2-\lambda; -\theta)$$

where $F_{2,2}(a_1, a_2; b_1, b_2; z)$ is a generalized hypergeometric function, defined as follows:

$$F_{2,2}(a_1, a_2; b_1, b_2; z) = \sum_{n=0}^{\infty} \frac{(a_1)_n(a_2)_n}{(b_1)_n(b_2)_n}\frac{z^n}{n!}$$

Here $(a)_n$ is the Pochhammer symbol or ascending factorial function, defined by $(a)_n = \Gamma(a+n)/\Gamma(a)$.

We can also use the following identity for the incomplete gamma function, $\gamma(x,z) = z^x x^{-1}F_{1,1}(x;x;-z)$. In terms of hypergeometric functions, the required inequality (21) can be re-arranged to give the following:

$$(1-z)e^{-\theta} < \left(\frac{1-\lambda}{2-\lambda}\right)\frac{F_{1,1}(1-\lambda; 2-\lambda; -\theta)F_{1,1}(2-\lambda; 3-\lambda; -\theta)}{F_{2,2}(1-\lambda, 1-\lambda; 2-\lambda, 2-\lambda; -\theta)} \tag{22}$$

In the limit as either $e^{-\theta} \to 0$ or $z \to 1$, this clearly always holds, since the right hand side is positive. This is consistent with the fact that in these limiting cases, $s_K^* = \lambda$, so $ds_K^*/d\lambda > 0$.

We first establish the following inequality for *any* $\theta \geq 0$ and $\lambda \in (0,1)$.

$$e^{-\theta} \leq \frac{F_{1,1}(1-\lambda; 2-\lambda; -\theta)F_{1,1}(2-\lambda; 3-\lambda; -\theta)}{F_{2,2}(1-\lambda, 1-\lambda; 2-\lambda, 2-\lambda; -\theta)} \tag{23}$$

By Kummer's first transformation formula, we have $F_{1,1}(b;c;z) = e^z F_{1,1}(c-b;c;-z)$. (See, for example, Andrews et al. (2000), [Eq. 4.1.11]). Substituting into (23), we need to show that

$$F_{2,2}(1-\lambda, 1-\lambda; 2-\lambda, 2-\lambda; -\theta) \leq e^{-\theta}F_{1,1}(1; 2-\lambda; \theta)F_{1,1}(1; 3-\lambda; \theta)$$

Now $F_{1,1}(1; 3-\lambda; \theta) \geq 1$ for all $\theta \geq 0$, since $F_{1,1}(1; 3-\lambda; 0) = 1$ and

$$\frac{\partial}{\partial x} F_{1,1}(b; c; x) = \frac{1}{c} \sum_{n=0}^{\infty} \frac{(b+1)_n}{(c+1)_n} \frac{x^n}{n!} \geq 0$$

So it suffices to show the following:

$$F_{2,2}(1-\lambda, 1-\lambda; 2-\lambda, 2-\lambda; -\theta) \leq e^{-\theta} F_{1,1}(1; 2-\lambda; \theta) \tag{24}$$

We start with the following equation, which is obtained from the more general equation (5.11) in Miller and Paris (2011) by specializing to $n = 1$. It can also be obtained from Prudnikov et al. (1990) [Eq. 7.12.1(2)], using Kummer's transformation for the $F_{1,1}$ function.

$$F_{2,2}(b, c; b+1, c+1; x) = e^x \frac{b}{b-c} \left[F_{1,1}(1; c+1; -x) - \frac{c}{b} F_{1,1}(1; 1+b; -x) \right]$$

Since we are interested in the case $b = c$, we need to take the limit of this expression as $b \to c$.

Letting $b = c + y$, we have

$$\begin{aligned}
& \lim_{b \to c} F_{2,2}(b, c; b+1, c+1; x) \\
= & \lim_{y \to 0} e^x \frac{c+y}{y} \left(F_{1,1}(1; c+1; -x) - \frac{c}{c+y} F_{1,1}(1; c+1+y; -x) \right) \\
= & \lim_{y \to 0} e^x (c+y) . \lim_{y \to 0} \left(\frac{F_{1,1}(1; c+1; -x) - \frac{c}{c+y} F_{1,1}(1; c+1+y; -x)}{y} \right) \\
= & \; ce^x . \lim_{y \to 0} \left(\frac{F_{1,1}(1; c+1; -x) - \frac{c}{c+y} F_{1,1}(1; c+1+y; -x)}{y} \right) \\
= & \; ce^x . \lim_{y \to 0} \left(\frac{(c+y) F_{1,1}(1; c+1; -x) - c F_{1,1}(1; c+1+y; -x)}{y\,(c+y)} \right) \\
= & \; ce^x . \left(\lim_{y \to 0} \frac{y F_{1,1}(1; c+1; -x)}{y(c+y)} + \lim_{y \to 0} \frac{c\,(F_{1,1}(1; c+1; -x) - F_{1,1}(1; c+1+y; -x))}{y(c+y)} \right) \\
= & \; e^x F_{1,1}(1; c+1; -x) + ce^x \lim_{y \to 0} \frac{F_{1,1}(1; c+1; -x) - F_{1,1}(1; c+1+y; -x)}{y}
\end{aligned}$$

The limit on the right is simply the partial derivative of the function $F_{1,1}(a_1; a_2; z)$ with respect to its second argument, evaluated at $a_1 = 1$, $a_2 = c + 1$ and $z = -x$. The form of this derivative found in Erdelyi et al. (1953) is the following:

$$\frac{\partial F_{1,1}(a_1;a_2;z)}{\partial a_2} = -\sum_{n=0}^{\infty} \frac{(a_1)_n}{(a_2)_n} \left(\psi(a_2+n) - \psi(a_2)\right) \frac{z^n}{n!}$$

where $\psi(a)$ is the digamma function, or psi function, defined by $\psi(a) = \Gamma'(a)/\Gamma(a)$. We therefore obtain:

$$F_{2,2}(c,c;c{+}1,c{+}1;-x) = e^x F_{1,1}(1;c{+}1;-x) - ce^x \sum_{n=0}^{\infty} \frac{1}{(c+1)_n} \left(\psi(c+1+n) - \psi(c+1)\right) \frac{(-x)^n}{n!}$$

Since $\psi(a)$ is strictly increasing for $a \geq 0$, we have $\frac{\partial F_{1,1}(a_1;a_2;z)}{\partial a_2} < 0$, so for any $c \geq 0$,

$$F_{2,2}(c,c;c+1,c+1;x) \leq e^x F_{1,1}(1;c+1;-x)$$

Letting $c = 1-\lambda$, we have $F_{2,2}(1-\lambda, 1-\lambda; 2-\lambda, 2-\lambda; -\theta) \leq e^{-\theta} F_{1,1}(1; 2-\lambda; \theta)$. So we have now shown (24) and therefore also (23).

Now suppose that $z > 1/(2-\lambda)$, so we have $1 - z < \frac{1-\lambda}{2-\lambda}$. Together with (23), this implies that the required inequality (22) holds. Hence we have $ds_K^*/d\lambda > 0$ provided that $z > 1/(2-\lambda)$.

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