A New-Monetarist Phillips Curve

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Abstract

Standard monetary models with labor search cannot generate the empirical correlations, in particular between unemployment and inflation, in both the short run and the long run. On the other hand, standard market-segmentation models cannot generate enough persistence, in particular of inflation. We propose a model with labor search and market segmentation, which can rightly complement each other to overcome these shortcomings. A temporary shock (short-run) to money growth causes the intensive-margin effect on employment, and reduces the unemployment rate and the nominal interest rate. A permanent shock (long-run) to money growth raises the unemployment rate and the nominal interest rate instead. Calibrated to the standard U.S. economy, shocks to money growth and productivity can generate enough persistence of inflation and unemployment in the correlation shown in the data, as a result of the extensive-margin effect on employment and market segmentation. Monetary shocks explain the volatility and persistence of nominal variables, but also their correlations with real variables. The optimal monetary policy minimizes the distortion from labor-market frictions, and eliminates the intertemporal monetary distortion and the market segmentation friction.

Keywords: Liquidity, Labor-market Search, Unemployment, Market Segmentation, Persistent Liquidity Effect, Persistent Inflation, Liquidity Trap, Optimal Monetary Policy, New Monetarism

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1 Introduction

This paper outlines a simple micro-founded model that features a persistent propagation mechanism on inflation, output, on both margins of employment, and on the nominal interest rate after a one-time shock to money growth. The model also reveals the mechanism by which structural relationships between a nominal variable and a real variable reverse as a short-run monetary policy turns into a long-run one. We analyze how the structural relationship, for example the liquidity effect and the "slope" of Phillips curve, depends on the characteristics of monetary policy. Having characterized the theoretical properties, namely the existence and uniqueness of the equilibrium and the comparative statics, we show the simple model can also generate the persistent inflation and unemployment with the correlation shown in the data, given the money growth and productivity featured in the U.S. data.

Our model is close to the seminal works of Shi (1998) and Berentsen, Menzio and Wright (2011, BMW), which also incorporate labor-market search in a new-monetarist environment[1]. Their models are elegantly constructed so as to provide clear understanding on how different frictions interact, by avoiding the complicated distribution effect. In Shi (1998), sellers maintain a positive level of inventory due to random matching in the good market. Assuming uniqueness of the steady state, Shi (1998) shows an U-shaped relationship between money growth (x-axis) and unemployment (y-axis) in the long run. In the calibration exercise, Shi (1998) shows output and employment persistently above the steady state after a one-time shock to money growth (which matches the high-frequency data), because of the dynamics of the inventory level and the search-inducing effect in the goods market. However, after a one-time shock to money growth, the model predicts that the nominal interest rate increases, and the effect on the inflation rate immediately returns to the previous level. In sum, Shi (1998) can capture rightly the short-run behaviors of the real variables but not the "liquidity effect" nor persistent inflation. On the other hand, emphasizing the hold-up problem as in Lagos and Wright (2005), BMW finds that the intertemporal distortion from using money features a negative inflation-output relationship. Together with the labor-market search, this implies that output and employment always decrease[2] after a one-time shock to money growth (which matches the low-frequency data). Like Shi (1998), BMW does not feature the liquidity effect nor the persistent inflation. Altogether, as pointed out in their paper, BMW can capture dynamics of variables in the long run but not

[1] See Mortensen and Pissaride (1994), Andofatto (1996) and Rogerson, Shimer and Wright (2005) for uses of search models in "macro-labor" literature. See Williamson and Wright (2010) for the detailed survey of the literautre of new monetarism. See Shi (2006) and Williamson and Wright (2009) for arguments on the importance of laying out key frictions in monetary economics.

[2] Non-neutrality of monetary policy in new monetarist models without unemployment can also be found in Berentsen, Camera and Waller (2006) and Milico (2006). Also see Lucas (1996) on this issue in the early literature. Rocheteau et al. (2008) and Dong (2010) use similar new monetarist environment, but instead indivisible labor model to capture unemployment. Depend on paramenters, they show inflation-unemployment relationship can be *either* positive or negative. We want to show the same new monetarist environment can yield *both:* negative in the short run and positive in the long run.

in the short run.

Our model can capture the stylized facts in both the short run and the long run because we explicitly model market segmentation and labor-market frictions. Theses stylized facts are illustrated in figure 4 and table 2, and we leave the details in the latter section. Liquidity is the level (in term of real goods) of medium of exchange available for transactions[3] like shopping or hiring. The liquidity effect in our model comes from market segmentation, related to ideas in Lucas (1990), Fuerst (1992), Christiano and Eichenbaum (1992), and many others. However the mechanism is different as it works through the intensive margin of employment. In the short run, the liquidity effect of monetary policy is that a temporary shock to money growth can increase firms' liquidity to pay for more labor service (intensive margin of employment) and search harder for workers (extensive margin of employment). This is done at the cost of reallocating liquidity from consumption with the help of market segmentation. The temporary shock to money growth also lowers the nominal interest rate. Such an ability to reallocate liquidity is not likely to be sustained when a temporary shock to money growth turns into permanent in the long run, as agents have longer time to neutralize the liquidity distortion. So, in the long run, the Fisher effect dominates and implies, instead, a permanent shock to money growth reduces the economy-wide liquidity, and the liquidity of firms in particular. Hence, both extensive and intensive margins decrease. Since inflation always moves in the direction of money growth, the model in the short run can feature a negative inflation-unemployment correlation, which can become positive in the long run. The quantitative performance of the model is summarized in table 6.

Another goal of our model is to feature the persistent liquidity effect on inflation and real variables. Our model features a persistent liquidity effect simply because the extensive margin of employment serves as an additional state variable, which provides persistent dynamics[4] under labor-market frictions of search and separation. The quantitative performance in generating persistence is summarized in table 5. There are also other models of liquidity effects, such as Alvarez, Lucas and Weber (2001), and Alvarez, Atkeson and Kehoe (2002). Of course unemployment is not their focus, hence no labor-market frictions are incorporated. In fact, these models are carefully designed so as to allow sharp characterization of the equilibrium. But the cost is that, a once-and-for-all increase in money growth factor raises inflation rate and reduces the nominal interest rate on impact, but then the inflation rate and the nominal interest rate return immediately to the previous level. For instance, in Lucas (1990), Christiano and Eichenbaum (1992), and Fuerst (1992), agents

[3] The notion of liquidity here follows Lucas (1990) and Williamson (2010), which is the intensive margin of medium of exchange. It can be interpreted as the amount of real value of money or of credit available per transaction. On the other hand, liquidity can also be defined as the extensive margin of medium of exchange, such as Lagos and Rocheatau (2008) and Wright (2011), which captures the likelihood that an object is accepted as the medium of exchange.

[4] For example, Andolfatto (1996) shows incorporating labor-market search to a standard RBC model improves the performance of matching the persistence of output growth shown in the data. Also see Wang and Shi (2006) for a quantitative result on incorporating search in both labor and good markets.

reunite in a "big household" to pool resources within a period. This shuts down any possible persistent effect. In Alvarez, Lucas and Weber (2001), a fraction $1-\lambda$ of the agents are permanently excluded from the asset market. In Alvarez, Atkeson and Kehoe (2002) accessing the asset market involves a fixed cost, which is varies from agent to agent. Yet all these market segmentation models are such that, at the end of the period the distribution of assets becomes degenerate. This mitigates the persistence of the propagation mechanism through the distribution of assets.

Exceptions to these are the new-monetarist model by Williamson (2006) and a deterministic version, the inventory-theoretic model of money demand developed by Alvarez, Atkeson and Edmond (2009, AAE). AAE studies an endowment economy with heterogeneous agents who have different remaining time, from 0 to $N-1$, to reallocate money from investment (in a brokerage account) to consumption (in a bank account). The models are constructed so as to contain the complexity of the distribution effect but the persistent liquidity effect is still featured. However, as Edmond and Weil (2008) point out, to match the data, the model needs to assume that agents can only access to their brokerage accounts for every $N = 38$ months. So a very long period of segmentation is needed to generate the persistent liquidity effect in these models[5]. Here we show that the labor-market friction can help to provide the needed persistence to match the data.

The analysis of this paper proceeds as follows. Section 2 first introduces the benchmark model without the persistence of matches in the labor market. This simplified model helps to understand the key frictions of the economy that result in various properties of a new-monetarist Phillips curve. It produces some useful propositions which helps to explain the simulation results in the latter section. We also study conditions for the existence and uniqueness of the monetary equilibrium. Then we move from the theoretical part to the quantitative part. Section 3 introduces the full model with persistence of matches in the labor markets. Then section 4 calibrates the full model to the standard U.S. economy. It illustrates a persistent propagation mechanism which matches the data in certain aspects. Section 5 characterizes properties of the optimal monetary policy in a general environment. Section 6 concludes and briefly compares the model with the monetarist models, cash-in-advanced models and new Keynesian models.

2 Benchmark Model without Persistence

Time is discrete and continues forever. There is continuum of households with unit mass, each consists of a continuum of consumers (she), workers (he) and firms (it), each with unit mass. Each period is divided

[5] Chiu (2007) shows the result of AAE may change qualitatively if N becomes endogenously determined instead. Khan and Thomas (2008) show incorporating idiosyncratic cost to participate the account can improve the model's performance in matching the inflation persistence.

into three sub-periods, where decentralized labor markets (LM), decentralized goods markets (GM), and centralized money markets (CM) sequentially take place. Money is needed for transaction in all three markets.

At the beginning of period, a typical household carries z_{t-1}^{P} real money from the last CM (in term of the price level in last period), which is to be distributed to firms. Also, each consumer is sent to GM with z_{t-1}^{C} real money, which cannot be changed during the period. Then an aggregate productivity A_t realizes. Then the central bank transfers Δ_t real money (in term of the current price level) to households. After the realization of Δ_t, households know the equilibrium price of money ϕ_t and the inflation factor $\pi_t \equiv \phi_{t-1}/\phi_t$.

After the money injection, each firm has $z_{t-1}^{P}/\pi_t + \Delta_t$ real money. Each firm searches for a worker in LM with intensity e_t. Each matched firm pays real wage $w_t \leq z_{t-1}^{P}/\pi_t + \Delta_t$ for having h_t labor service according to bargaining in LM, then goes to GM with the output. Unlike firms, each consumer takes the probability of matching with a firm in GM as given by n_t and no search effort is needed to match a firm. Each matched consumer pays real money $s_t \leq z_{t-1}^{C}$ for c_t goods according to bargaining in GM. Finally, agents return home with goods and money. Then each household trades $z_t^C + z_t^P$ goods in CM for money, where z_t^C will be distributed to consumers for shopping in the next GM. So the typical household brings z_t^P to the beginning of the next period. The above sequence repeats itself. Figure 1 summarizes the sequence of events within a period.

[Figure 1 inserts here]

In this simplified model, all matches are destroyed at the end of the period, so agents have to conduct a search again in the next period. Agents are anonymous. Lack of recordkeeping implies that money is needed for both transactions of consumption and production.

2.1 Preferences

Preferences of the typical household are given by

$$\mathbb{E}\sum_{t=1}^{\infty}\beta^{t-1}\left\{\int_{i\in[0,1]}U\left(c_t^i\right)di + X_t - \int_{i\in[0,1]}h_t^i di - \int_{i\in[0,1]}\kappa e_t^i di\right\}, \tag{1}$$

where $\beta \in (0,1)$ is the discount factor, $U\left(c_t^i\right)$ is the utility of the consumption of the i-th consumer in GM satisfying standard assumptions[6], X_t is the net consumption in CM as in Lagos and Wright (2005), $h_t^i \geq 0$ is the labor of the i-th worker, $e_t^i \geq 0$ is the search intensity of the i-th firm, and $\kappa > 0$ is the marginal disutility of search intensity. Each consumer consumes c_t^i on spot and then returns to the household. Each firm brings home the remaining output after the sale in GM. The remaining output is consumed as general

[6] As usual, we assume U is increasing and strictly concave function with $U(0) = 0$.

goods in CM as a part of X_t,. We assume the law of the large number applies, so any household is the same as the typical one.

2.2 Technology and Search

Firms produce perishable goods[7], where the output level in the LM of period t is

$$y_t = A_t f(h_t),$$

and A_t captures the stochastic productivity. The goods produced by the firms can be consumed by consumers in GM, or by that household in CM.

The good that is produced by the firms in a household is not consumed by any of the consumers in that household. The labor that is required by the firms in a household is not supplied by any of the workers in that household. So firms and consumers search for a bilateral trade in LM and GM respectively. In LM, anonymous firms decide the level of search intensity e that affects the probability of matching with an unemployed worker. The probability is given by $q(e)$ which satisfies standard assumptions[8]. Workers takes the probability of matching a firm as given as q_t. The unemployment rate is $u_t = 1 - q_t$. In GM, consumers takes the probability of matching a firm as given as n_t.

2.3 Production and Unemployment in the Labor Market

Profit maximization implies that the typical firm offers to hire h labor (intensive margin) with a real wage $w = h$ such that the worker always accepts: both accepting and rejecting the offer give zero surplus to the worker. We assume that firms (buyers in LM) can place a take-it-or-leave-it offer to workers. All our results are robust to general bargaining protocols[9]. The surplus of a firm holding z real money in LM is

$$\max_{e \geq 0} \{q(e) Y(z; A) - \kappa e\} + z,$$

[7] We assume output cannot be consumed by the worker and lasts until GM, so it cannot be used as commodity money.

[8] As usual, we assume $f(h)$ and $q(e)$ are increasing and strictly concave function with $f(0) = q(0) = 0$. The matching function $q(e)$ can be generalized to $q(e, v_0, u_0)$ to incorporate externality of labor market thickness, where u_0 and v_0 are measure of unmatched worker and firms at the begining of the period respectively. In our benchmark, we have $v_0 = u_0 = 1$ since all matches are destroyed at the end of period, so matching function is simply denoted as $q(e)$.

[9] All our results are robust since results are based on the necessary conditions of agent's optimal choice of money demand and search intensity only. In particular, the properties which is essential to prove our results are $Y_z \geq 0$, $Y_{zz} \leq 0$ and positive determinant of the associated Jacobian (for example see proof of proposition 1 to 4), which comes from the necessary conditions of maximization, rather than any particular assumption on the bargaining. As point out in Wright (2010), the protocol of bargaining does not matter in general because agents have endogenized the bargaining solution when they make the relevant decision on money demand (as in Wright (2010)) and search intensity (in our context). So for the ease of exposition, we stick to simplest bargaining protocol: take-it-or-leave-it.

where $Y(z;A)$ is the surplus of production from holding z real money:

$$Y(z;A) \equiv \max_{h\in[0,z]} \{Af(h) - h\}. \tag{2}$$

The simple form of production surplus $Y(z;A)$ comes from the feature that the firm has all the bargaining power in LM.

2.4 Consumption in the Good Market

The surplus of a consumer holding z real money in GM is

$$n_t V(z) + z,$$

where $V(z)$ is the surplus of consumption[10] from holding z real value of money:

$$V(z) = \max_{c\in[0,z]} \{U(c) - c\}. \tag{3}$$

We assume consumers (buyers in GM) can place a take-it-or-leave-it offer to firms. So a typical consumer offers to to buy c goods with at least $s = c$ real money such that the firm always accepts: both accepting and rejecting the offer give zero surplus to the firm in GM.

2.5 Money Demand in the Centralized Market and the Equilibrium

To model a price level and hence inflation, we derive the equilibrium nominal price of goods by allowing centralized trading of money in CM. Suppose a typical household trades goods X_t in CM with money at the nominal price ϕ_t^{-1}. At the end of period $t-1$, the typical household solves:

$$W^{CM} \equiv \max_{z_{t-1}^P, z_{t-1}^C} \left\{ -z_{t-1}^P - z_{t-1}^C + \beta \mathbb{E}_{t-1} \max_{e_t \geq 0} \left\{ \begin{array}{c} q(e_t) Y\left(\frac{z_{t-1}^P}{\pi_t} + \Delta_t; A_t\right) - \kappa e_t + \frac{z_{t-1}^P}{\pi_t} \\ + n_t V\left(\frac{z_{t-1}^C}{\pi_t}\right) + \frac{z_{t-1}^C}{\pi_t} + W^{CM} \end{array} \right\} \right\},$$

where $\pi_t = \phi_{t-1}/\phi_t$ is the gross inflation rate. The household arranges z_{t-1}^P real money for firms, which will be deflated to z_{t-1}^P/π_t in the LM in the next period. Firms also get the money injection Δ_t hence the total real money held by firms is $z_{t-1}^P/\pi_t + \Delta_t$.

In the equilibrium, the price of money ϕ_t induces the money demand equal to the money growth in CM, which implies

$$z_{t-1} = z_{t-1}^P + z_{t-1}^C,$$

[10] We assume parameters such that consumer's purchase is not bounded by the output level of firms.

where the level of real money is given by the central bank's budget:

$$z_t = \frac{z_{t-1}}{\pi_t} + \Delta_t,$$

where the left hand side is the total supply of real money in period t, which consists of the supply of real money z_{t-1}/π_t from the last period deflated by inflation, and the injection Δ_t at the beginning of period t. It implies that, in the equilibrium, the liquidity held by firms after the injection is

$$\frac{z_{t-1}^P}{\pi_t} + \Delta_t = z_t - \frac{z_{t-1}^C}{\pi_t}. \tag{4}$$

In the benchmark model, suppose the productivity A_t and the money growth factor $\mu_t \equiv M_t/M_{t-1} = 1 + \pi_t \Delta_t / z_{t-1}$ follow the standard formulation in the literature:

$$\log A_t = \log \overline{A} + \nu_t,$$

$$\log \mu_t = \log \overline{\mu} + \varepsilon_t,$$

where ν_t and ε_t are iid. We interpreted ε_t as the temporary/ temporary shock/ unanticipated component to the money growth rate in the short run, and $\overline{\mu}$ as the permanent/ permanent shock/ anticipated component of the money growth rate in the long run.

Consider a constant real balance of money $z = z_t$ and a constant real balance of consumption money $z^C = z_t^C$ for all t in the equilibrium, which implies the inflation rate equal to the money growth rate:

$$\pi_t = \mu_t.$$

At the symmetric equilibrium, each household trades zero net money in CM, hence the introduction of money trading in CM does not affect allocations. The matching probabilities in LM and GM at the symmetric equilibrium are

$$n_t = q_t = q\left(e_t\right) = 1 - u_t. \tag{5}$$

At the equilibrium, the first order condition of z_{t-1}^P and z_{t-1}^C are reduced to

$$\frac{1}{\beta} - \mathbb{E}_{t-1}\mu_t^{-1} = \mathbb{E}_{t-1}\left\{\frac{1-u_t}{\mu_t} Y_z\left(z - \frac{z^C}{\mu_t}\right)\right\}, \tag{6}$$

$$\frac{1}{\beta} - \mathbb{E}_{t-1}\mu_t^{-1} = \mathbb{E}_{t-1}\left\{\frac{1-u_t}{\mu_t} V_z\left(\frac{z^C}{\mu_t}\right)\right\}, \tag{7}$$

where $Y_z\left(z; A\right) \equiv \partial Y\left(z; A\right)/\partial z$ and $V_z\left(z\right) \equiv \partial V\left(z\right)/\partial z$.

At the equilibrium, the first order condition of h_t becomes

$$h_t = \min\{z - \frac{z^C}{\mu_t}, f_h^{-1}\left(A_t^{-1}\right)\}. \tag{8}$$

The first order condition of e_t becomes

$$\kappa = q_e\left(e_t\right) Y\left(z - \frac{z^C}{\mu_t}\right), \tag{9}$$

where $q_e\left(e\right)$ and $f_h\left(h\right)$ denote their first derivatives.

We assume parameters so as to support the equilibrium search intensity as an interior solution. This allows us to use the information in the second order conditions to establish results.

2.6 Equilibrium Nominal Interest Rate

To construct a price of production liquidity, we derive the equilibrium nominal interest rate in a hypothetical centralized bond market. Suppose that every household, after consumers are gone, can trade nominal bonds at the nominal price $\left(1+i_t\right)^{-1}$ at the beginning of the period (after it knows A_t and μ_t). This setting features limited participation to the bond market. Each nominal bond pays one unit of money at the end of the same period. At the symmetric equilibrium, each household trades zero net nominal bond, hence the original allocations is not affected. The equilibrium nominal interest rate is given by

$$i_t = q\left(e_t\right) Y_z\left(z - \frac{z^C}{\mu_t}\right). \tag{10}$$

Note that the equilibrium nominal interest rate is always non-negative, $i_t \geq 0$, otherwise, this implies $Af_h\left(z_{t-1}^P/\pi_t + \Delta_t\right) < 1$ hence the household can be better-off by leaving some money unused from production.

2.7 Existence and Uniqueness of Equilibrium

We need further restriction on parameters to guarantee the existence of equilibrium. Suppose A and μ are jointly distributed as $F\left(A,\mu\right)$. Define $\mathcal{J}^1 : \mathbb{R}_+ \times \left[0, \min_{\mu\in\Sigma_{A,\mu}} \mu\right] \times \Sigma_{A,\mu} \longrightarrow \mathbb{R}_+$ and $\mathcal{J}^2 : \mathbb{R}_+ \times \left[0, \min_{\mu\in\Sigma_{A,\mu}} \mu\right] \times \Sigma_{A,\mu} \longrightarrow \mathbb{R}_+$ as

$$\mathcal{J}^1\left(z,\alpha,A,\mu\right) \equiv \frac{\beta}{\mu}\left[q\left(\mathcal{E}\left(z,\alpha,A,\mu\right)\right) V_z\left(\frac{z\alpha}{\mu}\right) + 1\right], \tag{11}$$

$$\mathcal{J}^2\left(z,\alpha,A,\mu\right) \equiv \frac{\beta}{\mu}\left[q\left(\mathcal{E}\left(z,\alpha,A,\mu\right)\right) Y_z\left(z\left(1-\frac{\alpha}{\mu}\right);A\right) + 1\right]. \tag{12}$$

where $\mathcal{E}(z,\alpha,A,\mu)$ solves

$$\kappa = q_e(\mathcal{E})\, Y\left(z\left(1-\frac{\alpha}{\mu}\right);A\right). \tag{13}$$

We maintain the following assumptions on the derivative of $\mathcal{J}^1$ and $\mathcal{J}^2$:

Assumption A1 **(Existence)** There exists $z_1, z_2 \in \mathbb{R}_+$ and $\alpha_1 \leq \min_{\mu\in\Sigma_{A,\mu}} \mu$ such that

(a) $\mathbb{E}_{A,\mu}\mathcal{J}^2(z_1, 0, A, \mu) = 1$;

(b) $\mathbb{E}_{A,\mu}\mathcal{J}^2\left(z_2, \min_{\mu\in\Sigma_{A,\mu}} \mu, A, \mu\right) = 1$;

(c) $\mathbb{E}_{A,\mu}\mathcal{J}^1(z_2, \alpha_1, A, \mu) \leq 1$

Assumption A2 **(Uniqueness)** For all $z \in \mathbb{R}_+, \alpha \in \left[0, \min_{\mu\in\Sigma_{A,\mu}} \mu\right]$,

(a) $\mathbb{E}_{A,\mu}\mathcal{J}^1_z(z, \alpha, A, \mu) \leq 0$;

(b) $\mathbb{E}_{A,\mu}\mu^{-1} < 1/\beta$.

The existence and uniqueness of the intersection of the loci of $\mathbb{E}\mathcal{J}^1(z^*, \alpha^*, A, \mu) = 1$ and $\mathbb{E}\mathcal{J}^2(z^*, \alpha^*, A, \mu) = 1$ correspond to the existence and uniqueness of equilibrium with $\pi = \mu$, $z = z^*$ and $z^C = z^*\alpha^*$.

Part (a) and (b) of assumption A1 guarantees that the locus (z, α) on $\mathbb{E}_{A,\mu}\mathcal{J}^2(z, \alpha, A, \mu) = 1$ is well defined for all $\alpha \in \left[0, \min_{\mu\in\Sigma_{A,\mu}} \mu\right]$. The existence of such a locus is not always granted, for example, when $q(e)$ is linear and its slope is sufficiently less than κ. Part (c) of assumption A1 is not necessary for the existence of equilibrium, for example, if $\beta^{-1} = \mathbb{E}_{A,\mu}\mu^{-1}$ then equilibrium always exists as shown below, regardless part (c) is satisfied or not. Part (c) allows us to utilize the continuity property of related functions to prove the existence of the fixed point. Assumption A2 is not necessary true, which depends on the curvatures of U, f and q, so they need to be specified. A counter-example is when U is linear. In the equilibrium, an increase in liquidity has direct on the expected marginal surplus of consumers and that of firms. An increase in the value of money has a direct negative effect, since consumer can consume more, thus by the feature that marginal return is diminishing in liquidity, the expected marginal value of consumers decreases. Similarly, a higher value of money supports more hiring along the intensive margin h_t of employment, thus again by the feature of diminishing return, the expected marginal value of firms decreases.

Denote z_1^* is the infimum of z_1 solving $\mathbb{E}_{A,\mu}\mathcal{J}^2(z_1, 0, A, \mu) = 1$ and z_2^* is the supremum of z_2 solving $\mathbb{E}_{A,\mu}\mathcal{J}^2\left(z_1, \min_{\mu\in\Sigma_{A,\mu}} \mu, A, \mu\right) = 1$. The following proposition establishes the existence and uniqueness of a stationary monetary equilibrium.

Proposition 1 ***(Existence and Uniqueness of Monetary Equilibrium)*** *Given maintained assumptions.*

(a) For any F satisfying A1, there exists $z \in [z_1^*, z_2^*]$ and $\alpha \in \left[0, \min_{\mu \in \Sigma_{A,\mu}} \mu\right]$ satisfying (6) and (7);

(b) The solution is unique if A2 is satisfied as well.

(c) $\mathbb{E}_{A,\mu}\mu^{-1} \leq 1/\beta$ is a necessary condition for the existence.

(d) $\mathbb{E}_{A,\mu}\mu^{-1} = 1/\beta$ is a sufficient condition for multiple solutions.

Proof. See Appendix. ∎

Figure 2 illustrates how the monetary equilibrium is determined as the unique intersection of the loci of $\mathbb{E}\mathcal{J}^1(z^*, \alpha^*, A, \mu) = 1$ and $\mathbb{E}\mathcal{J}^2(z^*, \alpha^*, A, \mu) = 1$. The existence and uniqueness of the equilibrium when the central bank commits to long-run money growth factor is the special case where the distribution of μ becomes degenerate.

[Figure 2 inserts here]

Condition $\mathbb{E}_{A,\mu}\mu^{-1} \leq 1/\beta$ is the standard necessary condition for existence of a monetary equilibrium. In particular, it means that a household never finds keeping some money in LM and GM to buy good sin next CM optimal. If this is the case, the household will keep money unused forever, thus no money is circulated in the economy and no match happens.

Part (d) implies that a short-run monetary policy consistent with Friedman rule must lead to multiple monetary equilibria. We focus on monetary equilibrium, where monetary is used. Since there always exists an equilibrium where money is not used, hence the uniqueness of equilibrium is hopeless in that sense. To see how the condition $\mathbb{E}_{A,\mu}\mu^{-1} = 1/\beta$ is a sufficient condition for multiple monetary equilibria, suppose so. Then the first order conditions of M^P and M^C become

$$0 = \mathbb{E}\left\{\frac{q(e)}{\mu} Y_z\left(z^*\left(1 - \frac{\alpha^*}{\mu}\right)\right)\right\}, 0 = \mathbb{E}\left\{\frac{q(e)}{\mu} V_z\left(\frac{z^*\alpha^*}{\mu}\right)\right\},$$

which are satisfied if

$$Y_z\left(z^*\left(1 - \frac{\alpha^*}{\mu}\right)\right) = V_z\left(\frac{z^*\alpha^*}{\mu}\right) = 0.$$

Fixed any $\alpha^* < \min_{\mu \in \Sigma_{A,\mu}} \mu$, then the above holds for any sufficiently large z^*. So there are multiple solutions (z^*, α^*) to $\mathbb{E}\mathcal{J}^1(z^*, \alpha^*, A, \mu) = 1$ and $\mathbb{E}\mathcal{J}^2(z^*, \alpha^*, A, \mu) = 1$. In sum, the value of money is indeterminate under $\mathbb{E}_{A,\mu}\mu^{-1} = 1/\beta$, because the cash-in-advance constraints for firms and consumers are not binding. While it is the nominal variables, namely z and α, that have multiple equilibrium values, the underlying values of real variables, namely u, h and c, are unique.

2.8 The Short Run

Having provided sufficient conditions for the existence and uniqueness of the monetary equilibrium, we turn to characterize properties of the equilibrium. A short-run money injection raises the money growth factor μ_t. At the equilibrium, combining the first order conditions of the nominal bonds (10) and of the production money (6) implies:

$$\frac{1}{\beta} = \mathbb{E}_{t-1}\left\{\frac{1+i_t}{\pi_t}\right\}, \tag{14}$$

which is the (uncovered) Fisher equation.

The levels of z and z^C are independent to realization of μ_t and A_t. This should be interpreted as that any temporary shocks to money growth factor and to productivity in the short run do not change the fundamental of an economy. So short-run fluctuations on e_t and i_t driven by various μ_t and A_t realized are given equilibrium inflation, (16) and (17):

$$\pi_t = \overline{\mu}\exp(\varepsilon_t), \tag{15}$$

$$\kappa = q_e(e_t)\,Y\left(z - \frac{z^C}{\overline{\mu}}\exp(-\varepsilon_t); \overline{A}\exp(\nu_t)\right), \tag{16}$$

$$i_t = q(e_t)\,Y_z\left(z - \frac{z^C}{\overline{\mu}}\exp(-\varepsilon_t); \overline{A}\exp(\nu_t)\right). \tag{17}$$

The short-run unemployment rate u_t, the intensive margin of employment h_t given by (8) and output are

$$u_t = 1 - q(e_t), \tag{18}$$

$$h_t = \min\left\{z - \frac{z^C}{\overline{\mu}}\exp(-\varepsilon_t), f_h^{-1}\left(\frac{\exp(-\nu_t)}{\overline{A}}\right)\right\},$$

$$y_t = (1-u_t)\,A_t f(h_t).$$

The following proposition characterizes equilibrium relationships between a shock and a variable, ie, the impulse responses to shocks.

Proposition 2 ***(Short-Run)*** *Given the productivity shock ν_t (Given the temporary shock ε_t to money growth), higher ε_t (higher productivity ν_t) in the short run*

(a) reduces the unemployment rate u_t, raises output y_t and both intensive and extensive margins of employment, h_t and $q(e_t)$;

(b) reduces (raises) the nominal interest rate i_t;

(c) raises (no effect on) the equilibrium inflation rate π_t.

Proof. See Appendix. ■

Firms have higher liquidity $z - z^C/\mu_t$ reallocated from consumers after a one-time temporary shock ε_t to money growth. It is because the consumption liquidity z^C/μ_t is deflated by ε_t through inflation. So firms can buy more labor h_t. The surplus of firm increases and a higher search intensity e_t is justified. This implies that more firms get matched and higher output y_t. These effects are summarized in the proposition 2(a).

The liquidity effect, which is captured by the proposition 2 (b), comes from the intensive-margin effect on the employment. The Fisher equation also fails to hold ex post. In the equilibrium, the nominal interest rate i_t equates to the expected marginal surplus of firms for holding additional money, $q(e_t) Y_z(h_t)$. On the one hand, from the proposition 2(a), a temporary shock ε_t to money growth raises the extensive margin $q(e_t)$. On the other hand, ε_t also raises the intensive margin h_t and hence lowers the marginal product $Y_z(h_t)$ due to a concave production function. What the part (b) predicts is that the intensive-margin effect always dominates the extensive-margin effect. This result comes from the property of value maximization, in particular the second order conditions of maximization. So it generate a liquidity effect.

Structural relationships of variables are given by (16) to (18), which characterize a new-Monetarist Phillips curve. This establishes the equilibrium connections amongst the nominal interest rate, both margins of employment e_t and h_t, output y_t and inflation π_t. The results of proposition 2 cannot be interpreted as an the existence of a "trade-off" between inflation and unemployment in the structural sense, since a raise in inflation does not actually cause a drop in unemployment per se. Rather, they are all endogenous variables depending on the fundamentals, which include the distribution of shocks and their realizations.

We want to stress that the seemingly equivalence between the inflation rate and the money growth factor, that is $\pi_t = \mu_t$, is the equilibrium condition without persistent matches in the labor market. This is not the case in general, as it is clear later that the equilibrium condition in general is $\pi_t = \mu_t z_{t-1}/z_t$. Since z_t is a constant under the benchmark model, we have $\pi_t = \mu_t$. This is no longer the case when we introduce persistent matches in the labor market in the full model.

2.8.1 Liquidity Trap

We also note that the economy features a liquidity trap when the temporary shock ε_t to the money growth rate is sufficiently high, or the productivity shock ν_t is sufficiently low, such that the nominal interest rate becomes zero. During the liquidity trap, some money is hoarded rather fully used for hiring labor nor consumption. It is because, when high ε_t or low ν_t is realized, the transactional value of money drops, due to high inflation rate or low productivity for hiring such that the intensive margin of employment h_t in (8) takes the value of the second term in the min operator. At this extreme case, firms do not want to get any additional money in the bond market even the nominal interest rate is zero. So the money can not be drained and can only be hoarded.

2.8.2 Monetary Policy and New-Monetarist Phillips Curve

Having characterized equilibrium relationships between a shock and a variable, we turn to analyze how structural relationships depends on the distribution of money growth factors μ_t. The distribution of μ_t characterizes the monetary policy in the short run. Recall the short-run spurious "trade-off" between inflation and employment is given by:

$$h_t = \min\{z - \frac{z^C}{\pi_t}, f_h^{-1}(A_t)\},$$

which depends on the equilibrium liquidity z and the consumption liquidity z^C. The "slope" of a traditional Phillips curve, which is interpreted as the "trade-off" between inflation and intensive margin of employment, is z^C/π_t^2 if $h_t \neq f_h^1(A_t)$ and zero otherwise. To see the structural effect of a monetary policy, we need the following lemma:

Lemma 1 *Given maintained assumptions, then* $\mathbb{E}_{A_t,\mu_t}\mathcal{J}^2(z,\alpha,A_t,\mu_t)$ *is decreasing in* $\overline{\mu}$.

Proof. See Appendix. ■

Recall $\mathbb{E}_{A,\mu}\mathcal{J}^2(z,\alpha,A,\mu)$ is the ex-ante return of the production money in CM. Lemma 2 shows that the ex-ante return of the production money is lower in the equilibrium under a higher trend $\overline{\mu}$ of the money growth.

The effect of the trend of the money growth rate on the structural relationships can be illustrated by figure 2, which shows how $z = z^*$ and $z^C = z^*\alpha^*$ are determined as the intersection of $\mathbb{E}_{A,\mu}\mathcal{J}^2(z^*,\alpha^*,A,\mu) = 1$ and $\mathbb{E}_{A,\mu}\mathcal{J}^2(z^*,\alpha^*,A,\mu) = 1$. Recall $\mathbb{E}_{A,\mu}\mathcal{J}^2(z,\alpha,A,\mu) = 1$ is a upward sloping curve in $\alpha - z$ dimension, and $\mathbb{E}_{A,\mu}\mathcal{J}^1(z,\alpha,A,\mu) = 1$ slopes downward. When the money growth rate has a higher trend, which is captured by a higher $\overline{\mu}$, lemma 1 implies that $\mathbb{E}_{A,\mu}\mathcal{J}^2(z,\alpha,A,\mu) = 1$ shifts down. If the effect from the production liquidity dominates such that the effect on shifting $\mathbb{E}_{A,\mu}\mathcal{J}^1(z,\alpha,A,\mu) = 1$ becomes negligible, then a new intersection with lower z^* and higher α^* will be illustrated in figure 2. It means the equilibrium under a more volatile money growth factor will result in a lower real value of money $z^* = z$ and a higher consumption share of liquidity $\alpha^* = z^C/z$, since money becomes less desirable as a medium of exchange for production. If the drop of z^* dominates the rise of α^*, which again maybe because the effect from the production liquidity dominates, then the consumption liquidity $z^C = z^*\alpha^*$ follows the economy liquidity z to decrease as well.

In sum, the slope of Phillips curve z^C/π_t^2 decreases, hence the spurious inflation-employment trade-off becomes less exploitable under a higher trend of money growth.

The above result, that the short-run equilibrium "trade-off" between employment and inflation is not invariant to monetary policy, is a reminder of the Lucas critique. In particular, Lucas (1973) raises a point that, if the trade-off between employment and inflation is structural, then we would observe that countries with more volatile inflation also feature more volatile output, which is not the case. This fact is explained by a new-monetarist Phillips curve. In particular, a new-monetarist Phillips curve explicitly states that the slope of the observed "trade-off" depends on z^C and that its intercept depends on z. The values of z^C and z depend on the distribution of the short-run money growth factors. It is because the money demands, described by (6) and (7), are affected by the expectation of money growth factor, which depends on its distribution. The money demands for production and consumption determine z and z^C. In other words, the deeper structure behind the observed "trade-off" bases on how liquidity affects the economy, which depends on the characteristics of monetary policy and other fundamentals.

We also show an example that the observed trade-off between inflation and unemployment is indeed driven by the productivity shock under a endogenous monetary policy rule. Consider ε_t is no longer exogenous and the central bank follows a simple interest rate rule, which can be done by accommodating the money growth according to (17) in order to target the nominal interest rate at some level i^*. Suppose a positive productivity shock ν_t realizes, hence the nominal interest rate raises, by proposition 2. Then a higher ε_t is needed to maintain the nominal interest rate constant at i^*, also by proposition 2. Then the equilibrium inflation raises as well. Plotting the locus of the equilibrium inflation π_t against the equilibrium unemployment rate u_t with respect to different realization of A_t will also result in a downward-sloping Phillips curve. The entire Phillips curve and its shape depends on the distribution of A_t. However, it is obvious that none of the movements along the Phillips curve are driven by the trade-off of inflation.

2.9 The Long Run

The long-run analysis is to study how variables are affected when the trend component of productivity and of the money growth factor change. From lemma 1, we know the comparative statics could be ambiguous since the existence of productivity shock ν_t and the temporary shocks ε_t to the money growth factor will blur the whole picture. To highlight the long-run effect in this section, we assume $\nu_t = \varepsilon_t = 0$, ie, the short-run fluctuation channel is shut down.

At the long-run equilibrium, combining the first order conditions of nominal bonds and of production money implies:

$$\frac{1}{\beta} = \frac{1+i}{\overline{\mu}}, \tag{19}$$

which is the covered Fisher equation. Combining the first order conditions of consumption liquidity and that of production liquidity implies:

$$V_z\left(\frac{z^C}{\overline{\mu}}\right)=Y_z\left(z-\frac{z^C}{\overline{\mu}},\overline{A}\right), \tag{20}$$

As in the short run, the economy in the long run can be characterized by the equilibrium first order conditions of search intensity and that of nominal bond:

$$\kappa=q_e\left(e\right)Y\left(z-\frac{z^C}{\overline{\mu}},A\right), \tag{21}$$

$$\frac{\mu}{\beta}-1=q\left(e\right)Y_z\left(z-\frac{z^C}{\overline{\mu}},A\right), \tag{22}$$

and the long-run unemployment rate and output are

$$u=1-q\left(e\right), \tag{23}$$

$$y=\left(1-u\right)Af\left(z-\frac{z^C}{\overline{\mu}}\right).$$

The following proposition contrasts the opposite effect of money growth factor in the long run with the effect in the short run:

Proposition 3 ***(Long-Run)*** *Under the maintained assumption, a higher trend $\overline{\mu}$ of the money growth factor (higher $\overline{A}$)*

(a) raises (lowers) the long-run unemployment rate u, reduces (raises) output y_t and both intensive and extensive margins of employment, h and $q\left(e\right)$;

(b) raises the nominal interest rate i and inflation rate π;

(c) reduces (raises) economy liquidity z, consumption liquidity $z^C/\overline{\mu}$ and production liquidity $z-z^C/\overline{\mu}$

Proof. See Appendix. ■

Figure 3 illustrates how the search intensity e and the production liquidity $z-z^C/\overline{\mu}$ are determined in the long run as the intersection of the first order conditions of search intensity (21) and that of the nominal bond demand (22). The second order necessary condition of search intensity and that of nominal bond demand implies the slope of the first order condition of search intensity is flatter than that of nominal bond demand at the intersection. The left panel of figure 1 illustrates the effect of an increase in κ, which shifts down the first order condition of search intensity and leads to a lower e and a lower $z-z^C/\overline{\mu}$. If the order of slopes were reversed, then a higher search cost κ would instead raise the equilibrium search intensity e, which is the case of *preferences minimization.* This is ruled out by the second order necessary conditions.

[Figure 3 inserts here]

	Short-run		Long-run	
Variables\shocks	$\varepsilon_t\uparrow$	$\nu_t\uparrow$	$\overline{\mu}\uparrow$	$\overline{A}\uparrow$
Unemployment Rate u_t	$\downarrow$	$\downarrow$	$\uparrow$	$\downarrow$
Output y_t	$\uparrow$	$\uparrow$	$\downarrow$	$\uparrow$
Inflation Rate π_t	$\uparrow$	$-$	$\uparrow$	$-$
Nominal Interest Rate i_t	$\downarrow$	$\uparrow$	$\uparrow$	$\uparrow$
Extensive Margin $q(e_t)$	$\uparrow$	$\uparrow$	$\downarrow$	$\uparrow$
Intensive Margin h_t	$\uparrow$	$\uparrow$	$\downarrow$	$\uparrow$
Real value of money z	$-$	$-$	$\downarrow$	$\uparrow$
Production Liquidity $z - z^C/\pi_t$	$\uparrow$	$-$	$\downarrow$	$\uparrow$
Consumption Liquidity z^C/π_t	$\downarrow$	$-$	$\downarrow$	$\uparrow$

Table 1 Summary of comparative statics.

The order of slopes resulted from the second order necessary condition helps to establish proposition 3. A higher trend $\overline{\mu}$ of the money growth factor shifts up (22) in figure 3, and leads to a lower e and a lower $z - z^C/\overline{\mu}$. By the Fisher equation, such an increase in μ leads to a higher nominal interest rate i, a higher nominal bond demand, and hence a lower demand for the real balance of production money $z - z^C/\overline{\mu}$ which is an alternative for the nominal bond. Along (21) in figure 3, the equilibrium search intensity e decreases as the surplus of production drops. Table 1 summarizes all the results in the short run and in the long run.

The model predicts that the effects of a temporary shock to money growth reverse when it turns into permanent in the long run. The model implies that, on the one hand, the inflation-unemployment relationship can be negative in the short run. On the other hand, the inflation-unemployment relationship can be positive in the long run. This feature is consistent with the U.S. data, which is shown in Figure 4, even if we take the monetary shocks as the only driving force of the correlations between the unemployment rate and the inflation rate. Further quantitative assessments of the model are performed in the next section.

[Figure 4 inserts here]

3 Full Model with Propagation Mechanism

In this section we describe the full model where matches in the labor market are persistent. Then we calibrate the full model to the standard U.S. economy. See Arouba, et al (2010) for an example for calibrating a new-monetarist model to the data. The key result is that shocks to the productivity and the money growth factor can generate enough persistence on the inflation rate and others, as well as capture the correlation among variables shown in the data.

3.1 Persistence of Matches in the Labor Market

Consider that there is an idiosyncratic probability σ in the current period that the firm which matched a worker in the last period is separated from the worker forever. We follow the convention in the literature, for example Shimer (2005), that firms hit by separation shocks have to skip the current labor market. The evolution of the unemployment rate becomes $u_t = (1-q_t)\,u_{t-1} + \sigma\,(1-u_{t-1})$.

3.2 Equilibrium

For simplicity we assume households cannot distinguish the matched firms and the vacant firms, so they are given the same level of real money z^P_{t-1}. A monetary equilibrium in full model has similar mathematical structure as the one of the benchmark model[11]. At the end of period $t-1$, the typical household solves the functional equation:

$$W^{CM}_{t-1}(u_{t-1}) \equiv \max_{z^P_{t-1}, z^C_{t-1}} \left\{ -z^P_{t-1} - z^C_{t-1} + \beta \mathbb{E}_{t-1} \max_{e_t \geq 0} \left\{ \begin{array}{c} (1-u_t)\,Y\left(\frac{z^P_{t-1}}{\pi_t} + \Delta_t; A_t\right) + \frac{z^P_{t-1}}{\pi_t} - u_{t-1}\kappa e_t \\ +n_t V\left(\frac{z^C_{t-1}}{\pi_t}\right) + \frac{z^C_{t-1}}{\pi_t} + W^{CM}_t(u_t) \end{array} \right\} \right\},$$

where u_t is the state variable, which also happens to be the measure of vacant firm, with evolution according to

$$u_t = [1 - q(e_t)]\,u_{t-1} + \sigma\,(1-u_{t-1}). \tag{24}$$

At the equilibrium, the money supply equates the money demand, hence $z_{t-1} = z^P_{t-1} + z^C_{t-1}$, where the central bank's budget is

$$z_t = \frac{z_{t-1}}{\pi_t} + \Delta_t = \mu_t \frac{z_{t-1}}{\pi_t}.$$

As in the benchmark model, matches between a consumer and a firm are always separated after GM and every consumer has to shop for a new firm in the GM. Every consumer takes the probability of matching a firm in GM as given. At the symmetric equilibrium, $q_t = q(e_t)$ implies $n_t = 1 - u_t$.

The equilibrium is characterized by a system of first order conditions. The equilibrium value of money z_t and the real money of consumption z^C_{t-1} are determined by the first order conditions of z^P_{t-1} and z^C_{t-1} in the equilibrium, which are reduced to

$$1 = \beta \mathbb{E}_{t-1}\left[\frac{1}{\pi_t}\left\{(1-u_t)\,Y_z\left(z_t - \frac{z^C_{t-1}}{\pi_t}; A_t\right) + 1\right\}\right], \tag{25}$$

$$1 = \beta \mathbb{E}_{t-1}\left[\frac{1}{\pi_t}\left\{(1-u_t)\,V_z\left(\frac{z^C_{t-1}}{\pi_t}\right) + 1\right\}\right]. \tag{26}$$

[11] It is not straight-forward to prove the existence and uniqueness of stationary monetary equilibrium in the complete model. Under stochastic shock and additional dynamics of N_t, one can extend the assumption in proposition 1 to establish the existence of z_t and z^C_t for any given N_t. However, to show uniqueness, one has to prove the stochastic dynamic system is determinate, which is out of the scope here. In principle, the system can be reduced to four state variables, namely N_t, z_t, z^C_t and Γ_t, where the latter three of them are jump variables. So the system is determinate if there are exactly three underlying eigenvalues greater than unity. Otherwise, if the number of eigenvalue which is greater than unity is less than three, then the system become indeterminate, ie, there is a continuum of equilibrium path. If the number of eigenvalue which is greater than unity is exactly four, then the stationary monetary equilibrium is never attainable under shocks.

The first order condition of e_t (contingent on A_t and Δ_t) in the equilibrium is

$$\kappa = q_e(e_t)\left[Y\left(z_t - \frac{z_{t-1}^C}{\pi_t}; A_t\right) + \beta\Gamma_t\right], \tag{27}$$

where, by the envelope theorem, Γ_t is the expected surplus difference between a filled firm and a vacant firm, with the effective discounted factor $\beta[1-\sigma-q(e_{t+1})]$:

$$\Gamma_t \equiv \mathbb{E}_t\left\{\kappa e_{t+1} + [1-\sigma-q(e_{t+1})]\left[Y\left(z_{t+1} - \frac{z_t^C}{\pi_{t+1}}; A_{t+1}\right) + \beta\Gamma_{t+1}\right]\right\}.$$

3.2.1 Equilibrium Dynamics for Simulation

We summarize the equilibrium dynamics of the unemployment rate u_t, the output y_t, the inflation rate π_t and the nominal interest rate i_t for the use of calibration. The equilibrium dynamics of the unemployment rate and output are given by

$$u_t = [1-q(e_t)]u_{t-1} + \sigma(1-u_{t-1}), \tag{28}$$

$$y_t = (1-u_t)A_t f\left(z_t - \frac{z_{t-1}^C}{\pi_t}\right), \tag{29}$$

where e_t, z_t, and z_{t-1}^C solve (25), (26) and (27). By the definition of z_t and the first condition of bond, the equilibrium inflation rate and the nominal interest rate are given by

$$\pi_t = \frac{z_{t-1}}{z_t}\mu_t, \tag{30}$$

$$i_t = (1-u_t)Y_z\left(z_t - \frac{z_{t-1}^C}{\pi_t}; A_t\right). \tag{31}$$

The equilibrium inflation rate is no longer equal to the short-run money growth rate in general, since the value of money z_t is fluctuating with respect to shocks.

4 Data, Calibration and Simulation

4.1 Data

We use monthly data for the U.S. covering the period 1964:Jan - 2007:Dec obtained from the FRED database maintained at the Federal Reserve Bank of St Louis. We do not include data after 2007 to avoid any possible structural change of economy during the financial crisis of 2008-2009. The details of data can be found in Appendix B.

For the purpose of computing business-cycle statistics, we first natural logarithms of all series such that volatilities of variables are comparable. Then, we apply filters to extract the cycle components, so that the persistence of filtered variables come from purely the endogenous propagation mechanism, rather than

		u	y	π	i
Standard deviation		0.1757	0.0239	0.0031	0.0099
Monthly autocorrelation		0.9863	0.9792	0.4431	0.9790
	u	1	-0.8001	-0.1987	0.1958
Correlation matrix	y	$-$	1	0.1385	-0.3641
	π	$-$	$-$	1	0.0934

Table 2: Summary statistics of monthly U.S. data

the trends generated from the persistence of the exogenous shocks. We apply the Hodrick and Prescott (1997) (HP) filter to all series except the money growth factor. We apply the unobserved-component filter, which is discussed shortly, to the money growth factor to decompose it into the permanent shock and the temporary shock to the money growth. Shimer (2005) suggests a HP smoothing parameter 10^5 to extract the short-and-medium-run cycle from quarterly series, which is about 100 times its standard value in the literature. For the monthly series, Ravn and Uhlig (2002) suggests a smoothing parameter 0.16×30^4. So to extract the short-and-medium-run cycle, we use a smoothing parameter 16×30^4.

The time series correspond to the following variables in the model. After the transformation detailed in the Appendix, the unemployment rate is captured by u_t, output by $y_t \equiv (1 - u_t) A_t f(h_t)$, the nominal interest rate by i_t, the inflation rate by π_t (We have taken log so these becomes rates). Table 2 reports the summary statistics of the filtered monthly U.S. data.

4.2 Shocks

We capture shocks to productivity A_t and money growth factor μ_t as follows. As in standard RBC models, the process of productivity follows

$$
\begin{aligned}
A_t &= \overline{A}_t \exp(\nu_t), \\
\overline{A}_t &= A_0^{1-\rho_A} A_{t-1}^{\rho_A},
\end{aligned}
$$

where ν_t is an iid noise, A_0 is the long-run value of A_t and ρ_A captures the persistence of productivity. Households learn the realization of ν_t only after the LM of time t.

The process of the money growth factor follows a stochastic-trend formulation

$$
\begin{aligned}
\mu_t &= \overline{\mu}_t \exp(\varepsilon_t), \\
\log \overline{\mu}_t &= \log \mu_0 + \alpha_\mu \tau_t,
\end{aligned}
$$

where ε_t is an iid noise which may be correlated with ν_t. The process of τ_t is the unobserved trend which

Calibrated parameters		Steady target	
production function	$Af(h) = 0.6537h^{0.7}$	real money balance	$z = 1$
utility function	$U(c) = 1.9364c^{0.5}$	saving rate	$1 - \frac{z^C}{z} = 0.071$
matching function	$q(e) = 0.6460e^{0.38}$	search intensity	$e = 1$
vacancy cost	$\kappa = 0.0178$	unemployment	$u = 0.05$

Table 3: Calibration of steady state

follows

$$\tau_t = \alpha_\tau \tau_{t-1} + \eta_t,$$

where η_t is independent to ν_t and ε_t. Households learn the trend τ_t in the CM of time $t-1$, but only know ε_t after firms participate in the LM of time t. We interpret η_t as the permanent shock to the money growth and ε_t as the temporary shock to the money growth.

4.3 Calibration

We use explicit functions $f(h) = h^{0.7}$ for the production function, $U(c) = U_0 c^{0.5}$ for the utility function, and $q(e) = q_0 e^{0.38}$ for the matching function. The probability of a separation shock is 0.034. The discount rate is given by $\beta = 0.96^{1/12}$. The estimation on the filtered productivity concludes that $\rho_A = 0.9600$ and the standard deviation of ν_t is 0.0063. The choice of these parameter values are in line with Shimer (2005) and common in the literature.

The maximum likelihood estimation concludes that $\alpha_\mu = 5.362 \times 10^{-3}$, $\alpha_\tau = 0.9881$, the standard deviations of ε_t and η_t are 0.0038 and 0.1539. We use $\mu_0 = 1.02^{1/12}$ such that the inflation rate is two percents in the steady state. We assume the correlation coefficient of ε_t and ν_t is 0.95.

We calibrate the remaining parameters of our models by matching certain long-run calibration targets obtained from post-war U.S. data, using the steady state of the full model derived in Appendix C. Since the values of the steady-state variables are independent to the processes of shocks except their long-run value, so models with different processes of shocks are calibrated to the same steady state, hence their simulations are comparable. Such comparisons are described shortly. Parameters are calibrated such that the deterministic steady state is normalized to $u = 0.05$, $z = 1$, $e = 1$ and $z^C/z = 0.929$. The consumption ratio $z^C/z = 0.929$ corresponds to the average private saving rate of 7.1% during the same period. These mean $A_0 = 0.6537$, $q_0 = 0.6460$, $U_0 = 1.9364$, and $\kappa = 0.0178$. Table 3 summarizes the calibration exercise.

4.4 Simulation

To understand the effect of shocks on the business cycle phenomena, we compare the full model with alternate models with only productivity shock ν_t, with only anticipated money growth shock η_t, with both anticipated and unanticipated money growth shock η_t and ε_t, and with all productivity and both money growth shocks but independent ν_t and ε_t. The second model is a modification of Berentson, Menzio and Wright (2011) in line with our environment but it is used to generate the cycle component instead of the trend component. They are referred as the P model, BMW model, M model and PM model for short.

We follow the standard methodology for computing model-based statistics. We solve each version of the model using a second-order approximation around their steady states. We then simulate the model 1,000 times for 2,100 periods each, and eliminate the first 100 observations from each simulation. Then we apply the same filters to the simulated time series, as we apply to the data. Finally we compute the relevant statistics in each simulation, as we do on the data, and report the average of the statistics with the standard error across the simulations.

4.5 Impulse Response

Figure 5 to 8 illustrate the impulse response of shocks over time. The permanent shock η_t to money growth can be approximated by a long-run anticipated change in the money growth factor in the benchmark and the temporary shock ε_t to the money growth by a short-run unanticipated change in the money growth factor. Previous propositions in the benchmark model, summarized in table 1, help to explain these impulse response functions.

[Figure 5 to 8 inserts here]

The first panel of figure 5 (panel 6) show that a one-time increase in the permanent shock η_t to money growth leads to that the unemployment rate keep rising (output drops) for a few period and eventually decays (rises) to the steady state. These can be interpreted as results of proposition 3(a). A one-time increase in η_t changes the money growth factor permanently, so the real balance of money and the real money for firms in particular decrease, less labor service can be bought and less search intensity is put. As a result, there is higher unemployment rate and lower output. Similarly, a one-time increase in η_t pushes up the inflation rate and the nominal interest rate permanently, as in the long run the money growth rate must equates to the inflation rate, and the Fisher equation must hold. This explains the first panels of figure 7 and of figure 8.

The second panel of figure 5 (panel 6) show that a one-time increase in the temporary shock ε_t to money growth leads to that the unemployment rate first drops (output jumps) and eventually rises (drops)

Standard deviation	u	y	π	i
Data	.1757	.0239	.0031	.0099
Full model	.0206	.0680	.0041	.0091
P model	.0245	.0660	.0015	.0057
BMW model	.0001	.0086	.0038	.0035
M model	.0002	.0348	.0052	.0142
PM model	.0220	.0699	.0052	.0148

Table 4: Comparison of volatility

to the steady state. These can be interpreted as results of proposition 2(a). Due to market segmentation, A one-time increase in ε_t reallocates real balance from consumers to firms, so more labor service can be bought and more search intensity is put. As a result, there is lower unemployment rate and higher output. Similarly, a one-time increase in ε_t pushes up the inflation rate on spot, but remains above its steady state for awhile because of the extensive-margin effect. This explains the second panels of figure 7. The liquidity effect in the short run explains the drop of nominal interest rate in the second panel of figure 8.

The third panel of figure 5 (panel 6) show that a one-time increase in the productivity shock ν_t leads to that the unemployment rate first drops (output jumps) and eventually rises (drops) to the steady state. These can be interpreted as results of proposition 2(a). The nominal interest rate increases as the marginal product of labor increases, which explains the third panel of figure 7. A one-time increase in ν_t leads to a mean-reverting impulse response on the inflation rate: first drops then rises and decays toward the steady state from above. The inflation rate first drops because the real balance z_t increases, as the nominal interest rate in the next period i_{t+1} is expected to be higher. Then the real balance will decay toward the steady state from above, as the impulse response of the temporary shock ε_t to money growth, hence the inflation rate also decays from above. These explain the mean reverting impulse in the third panels of figure 7.

4.6 Volatility

Table 4 provides a summary regarding the volatility of variables. A stylized feature of the U.S. economy is that the unemployment rate is highly volatile, compared with other variables. Most of the observed fluctuation in total employment hours comes from the extensive margin rather than the intensive margin. This feature is replicated in the full model, not shown here, as the standard deviation of h_t generated is only 10% of the standard deviation of $1 - u_t$ generated.

The full model generates too little volatility of the unemployment rate but too much of output. Compared with the data, the models generates three times volatility of output, but only 12% volatility of the unemployment rate. This is in line with the findings by Andolfatto (1996) and Shimer (2005) in particular, which point out that a standard labor-search model can only generate less than 5% of the observed volatility of the unemployment rate.

Almost all the volatility of unemployment in the full model comes from the productivity shocks. The productivity shock generates about 100 times higher volatility of the unemployment rate in the P model than what the money growth shock generates in the M model. This is illustrated by the impulse responses functions on u_t in figure 5. It is because any change in the unemployment rate is induced from the change of the intensive margin: firms put higher search intensity for unemployed workers because the higher real money (reallocated from consumption) by the money growth shock can afford firms to buy more labor service. However such an effect from the intensive margin must be bounded by the feature of diminishing return of further labor service, as $f(h)$ is strictly concave. Also, the U.S. economy shows that the volatility of the productivity is 2 times higher than the volatility of the temporary shock to the money growth (the volatility contribution from the permanent shock is negligible). These two factors imply much less volatility of the unemployment rate can be generated by the money growth shocks than the productivity shock.

On the other hand, the full model can generate rightly the volatilities of the inflation rate and of the nominal interest rate. This is illustrated by the impulse responses functions on π_t and on i_t in figure 7 and figure 8 respectively. Most of volatilities of the inflation rate and of the nominal interest rate come from the money growth shocks. Compared the M model with the P model, the money growth shock generates about 3.5 times higher volatility of the inflation rate, and 2.5 times higher volatility of the nominal interest rate. The feature of low inflation volatility by productivity is highlighted in the benchmark model: productivity A_t can only affect inflation π_t by affecting the *growth* of the real balance z_t, which depends on the expected productivity and the number of matches in the future. Since productivity is persistent, any growth in the real balance z_t is predictable and hence mostly filtered out. So the productivity shock generates much less volatility of the inflation rate than the money growth shocks, though the productivity shock is much more volatile than the money supply shocks.

Compared the M model with the BMW model, a significant part of volatilities of output and the nominal interest rate comes from the temporary shock to the money growth rather than the permanent shock to the money growth. It is because of the strong liquidity effect shown in the data: the nominal interest rate, hence the intensive margin of employment, is sensitive, negatively, to the temporary shock to the money growth, as stated in the proposition 2.

Incorporating the productivity shocks and the money growth shocks together actually tends to cancel each other. The volatilities of variables are lower in the PM model. The volatilities are ever lower in the full model, where the two shocks are highly correlated.

Autocorrelation	u	y	π	i
Data	.9863	.9792	.4431	.9541
Full model	.9629	.7682	.5324	.1021
P model	.9726	.9723	$-.0300$	.0068
BMW model	.9869	.9846	.8477	.9846
M model	.3562	.0180	.3557	.0179
PM model	.9690	.7457	.3189	.0229

Table 5: Comparison of persistence

4.7 Persistence

Table 5 provides a summary regarding the persistence of variables. A stylized feature of the U.S. economy is that the propagation mechanism is highly persistent. This is reflected by the fact that even having applied the filter, variables still have high autocorrelation coefficients. The full models fails to generate enough persistence of the nominal interest rate, which is only about 11% persistence in the data. This aspect of result is in line with the search-inventory model of Shi (1998) without liquidity effect. The full model can only generate modest persistence of output. However, the full model performs well in matching the autocorrelation coefficients of the unemployment rate and the inflation rate.

A standard labor-search model with the productivity shocks usually can generate rightly the persistence of unemployment and output, due to the persistent dynamics (24) of unemployment. This result extends to the P model and BMW model. In both model, almost perfect autocorrelation coefficients of the unemployment rate and of output are generated, which are illustrated in figure 5 and 6. The BMW model can also generate a rightly level of persistence of the nominal interest rate as the data, although about 2 times persistence of the inflation rate as the data. On contrary, the P model fails to generate enough persistence of the inflation rate and the nominal interest rate. These are illustrated in figure 7 and 8. Most of the persistence of the nominal interest rate in the P model is due to the persistence of the productivity shocks, which is mostly filtered out. Also, the productivity shock generates a mean-reverting impulse response on the inflation rate, as shown in figure 5, so the P model features a negative autocorrelation coefficient of the inflation rate instead of a close-to-unity one. In sum, the BMW model predicts too high inflation persistence but the P model predicts too low. It explains why the full model, as a blending of two, can have a better capture of the inflation persistence in the data.

On the other hand, having incorporated the volatile temporary shock to the money growth, the M model generate much less persistence of the variables than the BMW model. This rightly corrects the too-high persistence of the inflation rate in the BMW model. However, the cost is that the persistence of the nominal interest rate generated in the M model becomes far from enough.

We also note that much higher persistence of the nominal interest rate and of the inflation rate are generated in the full model than in the PM model. The money growth shock and the productivity shocks are highly correlated in the full model. The higher persistence than the PM model maybe another side of the same coin of the lower volatility than the PM model. Since the effects of the productivity shock and the money growth shocks tend to cancel each other, variables becomes less volatile and more persistent when the shocks are highly correlated.

4.7.1 Discussion on Persistent Inflation

To understand the mechanism of persistent inflation in the full model, recall that the equilibrium inflation rate and the equilibrium real balance are given by

$$\pi_t = \frac{z_{t-1}}{z_t}\mu_t,$$

$$z_t = \mathbb{E}_t \left[\frac{\beta (1 + i_{t+1})}{\mu_{t+1}} z_{t+1} \right]. \tag{32}$$

A one-time positive temporary shock ε_t to the money growth raises the equilibrium inflation rate immediately. To capture a persistent inflation above its steady-state value after that, it requires that z_{t-1}/z_t is converging to unity from above. This implies z_t decays toward to its steady state from above. Intuitively, the persistently higher real value of money motivates households to persistently allocate more money to firms. So this implies a inflation rate persistently higher that its steady state.

A one-time positive temporary shock ε_t raises z_t persistently because of the extensive-margin effect. After a one-time positive temporary shock ε_t to the money growth, firms have liquidity reallocated from consumers due to the limited participation to money injection. The surplus of firm increases and a higher search intensity is justified. This implies that more firms get matched and are ready for production in the next period. There are two implications from the rise in the extensive margin of employment. First, production money becomes more worthy to spend as the household's continuum of firms has higher margin return for additional money on average. The fact that more workers employed in the continuum of firms works like a higher productivity of a representative firm. The labor market friction implies that it is not "free" to hire or fire a worker, so the rise in the extensive margin of employment, hence the productivity of a representative firm, remains in force for awhile. Second, the consumption money becomes more worthy to spend, since consumers are more likely to match a firms in LM, as the share of firms with output increases. Both implies money becomes more valuable, ie, z_t is higher than its steady state value after a one-time positive temporary shock ε_t.

Graphically, depicting (25) and (26) in any period after a one-time positive temporary shock ε_t a la

Correlation	$u\&y$	$u\&\pi$	$u\&i$	$y\&\pi$	$y\&i$	$\pi\&i$
Data	−.8001	−.1987	.1958	.1385	−.3641	.0934
Full model	−.9172	−.0430	.1830	.1779	−.5543	−.2875
P model	−.9941	.0613	−.2488	.0421	.1474	−.9788
BMW model	−.8789	.8176	.8789	−.9400	−1.000	.9400
M model	−.4024	−.1230	.4026	.5994	−1.000	−.5995
PM model	−.8705	.0748	−.0297	.2308	−.4571	−.6230

Table 6: Comparison of correlations

figure 2, then both curves in figure 2 shift up. This results in a higher intersection. It corresponds to a higher z_t in the full model.

Compared to the benchmark model, the effect on z_t is persistent since we have the unemployment rate u_t as the additional state variable in solving z_t in (25) and (26). The "persistence" of the unemployment rate is captured by its AR1 term $1 - \sigma - q(e_t)$. So the dynamics of u_t, and hence z_t, is more persistent when both $q(e_t)$ and σ are low. In other words, the effect of z_t is persistent as long as the share of the filled firms remain stable in LM, due to either that separation shocks do not frequently hit (outflow from labor market is low) or that the equilibrium search intensity is low (inflow to labor market is low), ie, matches in labor market are persistent. However, low values of $q(e_t)$ and σ also imply the unemployment rate is not volatile. This explains why the full model fails to get high volatility of the unemployment rate and high persistent of the inflation rate at the same time.

4.8 Structural Correlations

Table 6 provides a summary regarding the contemporaneous correlations of variables generated by shocks to productivity and money growth. The strong negative correlation between the unemployment rate u_t and output y_t in the data is obvious. The U.S. data also shows a negative correlation between u_t and the inflation rate π_t, a la the Phillips curve, and between y_t and the nominal interest rate i_t. The positive correlations is observed between u_t and i_t, between y_t and π_t, and between π_t and i_t thought the relationship is rather weak.

The full model fails to generate the positive correlation between π_t and i_t. Combine the results of impulse responses, only the permanent shock η_t to money growth can generate a positive correlation between π_t and i_t, as a result of the Fisher equation shown in the BMW model. It is dominated since the volatility of η_t is much lower. So the full model shows a negative correlation between the inflation rate and the nominal interest rate.

On the success side, the full model predicts the right correlations of the remaining variables as shown in the data. Furthermore, the full model can capture the exact correlations between u_t and i_t and between y_t and u_t as shown in the data. The full model generates a little stronger correlation between u_t and y_t and between y_t and i_t, and weaker correlation between u_t and π_t.

To understand why the full model performs well, we first look at the P model, the M model and the PM model. The P model illustrates the structural correlations driven by the productivity shock. The P model can only generate correct correlations between u_t and y_t and between y_t and π_t, although too strong for the former and too weak for the latter These two correct correlations, however, can also be generate by shocks to money growth in the M model but become too weak for the u_t/y_t correlation and too strong for the y_t/π_t correlation. So blending two shocks in the full model improves the performance. The rest of the correlation pairs except π_t/i_t one can be correctly predicted by the M model. So to strengthen the correlations produced by the M model, the full model needs a high correlation between the productivity shock ν_t and the temporary shock ε_t to money growth. These can be seen as simply blending ν_t and ε_t without correlation in the PM model cannot generate the negative correlation between u_t and π_t, nor the positive correlation between u_t and i_t.

4.9 Discussion on the Quantitative Results

A lesson from the quantitative exercise is that, the productivity shock plays a larger role in generating the volatility and persistence of the real variables, namely the unemployment rate, though still far from enough, and output. On the other hand, the monetary shocks play a larger role in generating the volatility and persistence of the nominal variables, namely the inflation rate and the nominal interest rate. The ability of matching the persistence of variables comes from the labor-market frictions. The diverse achievement in matching the second moments in the data implies that a combination of the productivity shock and monetary shocks is needed to capture the whole picture.

Furthermore, the monetary shocks also have an important role to generate all the correlations between a real variable and a nominal variable observed in the data. The success in matching the correlations between a real variable and a nominal variable highlights the importance of incorporating the market segmentation to generate the liquidity effect. The productivity shock can only rightly predict the positive correlation between output and the inflation rate.

5 Optimal Monetary Policy

Suppose the central bank can implement any path of money supply such that μ_t becomes endogenous. We extend the analysis of efficiency to general process of productivity shock, which is given by the following proposition:

Proposition 4 *Suppose the monetary equilibrium and the optimal policy exists, an optimal contingent money growth factor* $\{\mu_s^*\}_{s=t}^{\infty}$*, which maximizes* W_t^{CM}*, is*

$$\mu_t^* = \beta \frac{z_t^*}{z_{t-1}^*}. \tag{33}$$

where z_t^* *(depends on* A_t*) and* α^* *solve*

$$Y_z\left(z_t^* - \alpha^*; A_t\right) = V_z\left(\alpha^*\right) = 0. \tag{34}$$

Proof. See Appendix. ■

A monetary policy has three roles, namely to minimize the efficiency distortion by labor-search, to eliminate the intertemporal distortion due to using money as the medium of exchange, and to eliminate the intratemporal distortion due to the market segmentation. The three roles are actually consistent with each other. They boils down to target the zero nominal interest rate, which is the Friedman rule, even in a general environment. In the standard Lagos and Wright (2005) model with endogenous search intensity, such an efficiency distortion by search does not arise if the money holder (firms here) is also the searcher and have all the bargaining power. A version of Hosios rule is satisfied in that setting. However, in our model, that is no longer true, because of the additional consumer search on top of the labor search. The efficiency distortion by the labor search arises because firms do not endogenize the consumer's surplus when they choose the level of search intensity, hence it is always lower than the optimal level. The unemployment rate is always too high. In other words, the efficiency distortion by the labor search is only eliminated if firms can also place a take-it-or-leave-it offer to consumers as well. But then the monetary equilibrium collapses, as consumers never hold money in the equilibrium due to the hold-up problem. Hence, the efficiency distortion by the labor search is inevitable in the model. However it can be minimized by the optimal monetary policy. Since firms do not consider the consumer surplus in deciding the level of search intensity, so holding other things equal, encouraging higher search intensity always improve household welfare in the equilibrium, which can be done by increasing the equilibrium firm surplus. However, the firm surplus is the highest under zero nominal interest rate, where firms always get "free" money from the bond market, so the efficiency distortion by the labor search is minimized then.

On the other hand, when the central bank commits to zero nominal interest rate, firms always have enough production liquidity to hire labor at the desirable level. Otherwise there are some realizations of aggregate shocks where firms need more liquidity in LM. It will drive up the nominal interest rate to some positive level, which leads to contradiction. So the intertemporal distortion by the usage of money as medium of exchange is eliminated.

Lastly, since such commitment of zero nominal interest rate is full anticipated in CM, the ex post marginal money surplus of consumer, V_z, is always zero. Otherwise, if there are realizations of aggregate shocks where consumers need more liquidity in GM, it will drive the ex ante marginal money surplus of consumer to some positive level. Under the commitment of zero nominal interest rate, it implies the ex ant return of holding money for consumption is definitely higher for production, so all money goes to consumption until the level of consumption liquidity z_{t-1}^C is high enough to buy the desirable level of goods under any realization aggregate shocks. Hence this is the level where both consumption and production have the same ex post monetary return, which are zero. So the intratemporal distortion is also eliminated.

The issue left is how to commit zero nominal interest rate in the equilibrium by a monetary policy under general environment. Such an optimal money policy is stated in proposition 4. The central bank should adjust the money growth factor in line with the growth of the equilibrium real value of money, which is the inverted price stickiness. Under persistent aggregate shocks and persistent matches, the equilibrium real balance of money and hence the price stickiness are fluctuating. Under the optimal money growth factor, it means the central bank should inject more money when the price is sticky, and vice versa, such as to stabilize the equilibrium inflation rate at the real discount rate. This automatically targets the nominal interest rate at zero.

Proposition 4 also prescribes the optimal monetary policy to accommodate productivity shocks with the market segmentation. Suppose the productivity A_t rises unexpectedly. Then (34) implies z_t^* increases. Hence by the optimal rule of money growth (33), the central bank should rises the short-run money growth factor μ_t as well. It is because firms want more liquidity, but cannot do so due to the market segmentation. So it is optimal to inject liquidity to production through a money injection. Since the optimal rule of money growth (33) is fully anticipated, so it has no effect on the consumption liquidity. In sum, under the optimal monetary policy, the central bank always injects or drains enough liquidity to the economy such that the optimal allocation is always attained.

Note, as in Lagos (2010), the optimal monetary policy that implements a Friedman rule is not unique. In particular, fix a α^* solving $V_z(\alpha^*) = 0$, then any $\alpha \geq \alpha^*$ also solves $V_z(\alpha) = 0$. Similarly, given any A_t and any α^* solving $V_z(\alpha^*) = 0$ and fix a z_t^* solving $Y_z(z_t^* - \alpha^*; A_t) = 0$, then any $z_t \geq z_t^*$ also solves $Y_z(z_t - \alpha^*; A_t) = 0$. These imply the optimal monetary policy (33) is not unique. Furthermore, if the real value of money is bounded from above and hence $z^{**} = \sup_t z_t^*$ exists, then a Friedman rule can be implemented simply by setting a constant money growth factor $\mu_t = \beta$ such that $z_t = z^{**}$ for all t.

6 Discussion and Conclusion

A celebrated lesson from the criticism of Friedman (1968) and Phelps (1969) on the Phillips curve is that the central bank cannot keep surprising the pubic with unanticipated money injections indefinitely in order to maintain a low unemployment rate, so any exploitable trade-off between inflation and unemployment, if exists, ceases to exist in the long run. They conclude that in the long run, the economy should attain a constant natural rate of unemployment, and any permanent increase in money growth factor will only raise the inflation rate. Our model shows that a constant natural rate of unemployment is not immune to Lucas critique[12], since our micro-founded model of unemployment shows that, in the long run, raising the money growth factor can raise the unemployment rate as well. Actually a constant natural rate of unemployment is not the only scenario plausible under the dictum of no trade-off in the long run, a *positive* unemployment-inflation correlation in the long run is also plausible. This is also consistent with the empirical finding on the long-run relationship between inflation and unemployment in the U.S..

Another goal is to design a model that can feature persistent inflation, as well as both the negative inflation-unemployment relationship in the short run and the positive inflation-unemployment relationship in the long run as the full picture. A popular model that also builds on the microfoundation of the optimizing agents is New Keynesian Phillips curve (NKPC). A basic cashless NKPC has the reduced form[13]:

$$\pi_t = \beta \mathbb{E}_t \pi_{t+1} + \gamma y_t,$$

where π_t is the net inflation rate, $\beta < 1$ is the firm's discount rate, and y_t is the output gap between the real output under the Calvo setting of time-contingent pricing and the one under the flexible price. One criticism[14] is the lack of persistent inflation in NKPC due to the rational forward-looking behavior of the price-setting firms. Solutions are to assume "sticky information", or irrational firms with "backward-looking" behaviors so as to incorporate the lagging inflation terms in the NKPC. Our model illustrates that a persistent inflation can be featured without relying on the irrationality of firms or information frictions, once the frictions of search and separation, which are found naturally in many economic activities, is incorporated in the labor market.

How do we recognize the lack of persistence in standard models of money? Recall that, using notations

[12] The essence of Lucas critique (1976) on models for policy analysis, in our own words, is that any model should be deep enough such that fundamental economic structure where policy conclusion builds on should be invariant to monetary policy. Otherwise, the policy implication is not robust and deeper structure should be modeled. In the current context, the deep structure of the economy are frictions of labor-search and of limited recordkeeping, which are fundamental to unemployment and monetary exchange. The later founds the environment to study how change of nominal price over time, ie, inflation, is affected by monetary policy.

[13] See Ch 3 of Woodford (2003).

[14] See Rudd and Whelan (2007).

in our model, the inflation rate can always written as:

$$\pi_t = \frac{z_{t-1}}{z_t}\mu_t,$$

where the term z_{t-1}/z_t captures the price stickiness with respect to money growth. The class of flexible price models features $z_{t-1}/z_t = 1$ since the money velocity is constant. Then it is clear that persistent inflation is implausible in models of flexible prices: inflation returns immediately to the previous level after an once-and-for-all increase in money growth factor. If price is sticky such that $z_{t-1}/z_t < 1$ then the inflation rate is even mean-reverting after a one-time shock to money growth.

We show that the intensive-margin effect can generate a negative correlation between inflation and unemployment driven by the temporary shock to money growth, which is consistent with the correlations shown in the data. We also show in that a permanent shock to money growth leads to *lower* output and employment, which is consistent with the long-run results in Berentsen, Menzio and Wright (2011). The ability of generating the persistence of variables comes from the extensive-margin effect based on the labor-market frictions. The success in matching the correlations between a real variable and a nominal variable highlights the importance of incorporating the market segmentation to generate the liquidity effect through the intensive-margin effect.

Calibrated to the U.S. economy, the model predicts the persistent inflation and unemployment in the correlation shown in the data. A lesson from the quantitative exercise is that, the monetary shocks also have a dominating role to generate all the correlations between a real variable and a nominal variable observed in the data. The productivity shock can only rightly predict the positive correlation between output and the inflation rate. However, the productivity shock plays a larger role in generating the volatility and persistence of the real variables, and the monetary shocks play larger role in generating the volatility and persistence of the nominal variables, namely the inflation rate and the nominal interest rate. Incorporating endogenous money supply and credit channel may be promising directions to improve the quantitative results, especially to generate more a persistent nominal interest rate, to generate a stronger negative correlation between the unemployment rate and the inflation rate (ie, greater slope of a Phillips curve) and to generate a positive correlation between the nominal interest rate and the inflation rate. We leave the possibility for future research.

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7 Appendix A

7.1 Proof of Proposition 1

We first need the following lemma:

Lemma 2 *Under the maintained assumptions,* $E_{A,\mu}J^1(z,\alpha,A,\mu)$ *and* $E_{A,\mu}J^1(z,\alpha,A,\mu)$ *are continuous in* z *and* α*. Also*

(a) $E_{A,\mu}J^1_\alpha(z,\alpha,A,\mu)\leq 0$;

(b) $E_{A,\mu}J^2_z(z,\alpha,A,\mu)\leq 0$;

(c) $E_{A,\mu}J^2_\alpha(z,\alpha,A,\mu)\geq 0$.

Proof. First we show part (b) and (c). Recall he first order conditions of search intensity and nominal bond demand

$$\kappa = q_e(e)\,Y\left(z-\frac{\alpha}{\mu};A\right),$$

$$i = q(e)\, Y_z\left(z - \frac{\alpha}{\mu}; A\right),$$

where $Y(z)$ is defined as

$$Y(z) \equiv \max_{h \le z} \{Af(h) - h\}.$$

It is straight-forward to show that

$$Y_z(z) \ge 0, Y_A(z) \ge 0, Y_{zz}(z) \le 0, Y_{zA}(z) \ge 0.$$

Let denote the Jacobian $J\left(e, z - \frac{\alpha}{\mu}\right)$ of the above system of equations:

$$J\left(e, z - \frac{\alpha}{\mu}\right) \equiv \begin{pmatrix} q_{ee}(e)\, Y\left(z - \frac{\alpha}{\mu}\right) & q_e(e)\, Y_z\left(z - \frac{\alpha}{\mu}\right) \\ q_e(e)\, Y_z\left(z - \frac{\alpha}{\mu}\right) & q(e)\, Y_{zz}\left(z - \frac{\alpha}{\mu}\right) \end{pmatrix}.$$

The second order necessary condition in the equilibrium implies

$$\det\left(J\left(e, z - \frac{\alpha}{\mu}\right)\right) \ge 0,$$

Taking z and α as independent variables and i and e as dependent variables, then we have

$$\frac{\partial i}{\partial z} = \frac{\det\left(J\left(e, z - \frac{\alpha}{\mu}\right)\right)}{q_{ee}(e)\, Y\left(z - \frac{\alpha}{\mu}\right)} \le 0, \frac{\partial i}{\partial \alpha} = -\frac{1}{\mu} \frac{\det\left(J\left(e, z - \frac{\alpha}{\mu}\right)\right)}{q_{ee}(e)\, Y\left(z - \frac{\alpha}{\mu}\right)} \ge 0.$$

Denote the reduced dependent of i on z and α as $i(z, \alpha; A, \mu)$. Note that

$$\mathcal{J}^2(z, \alpha, A, \mu) = \frac{i(z, \alpha; A, \mu)}{\mu},$$

hence $J^2(z, \alpha, A, \mu)$ is decreasing in z and increasing in α. Thus we prove part (b) and (c).

We want to prove part (a). Note the above system also implies

$$\frac{\partial \mathcal{E}}{\partial \alpha} = \frac{1}{\mu} \frac{i q_e(e)^2}{\kappa q_{ee}(e)\, q(e)} \le 0.$$

Recall

$$\mathcal{J}^1(z, \alpha, A, \mu) \equiv \frac{q(\mathcal{E}(z, \alpha, A, \mu))}{\mu} V_z\left(\frac{z\alpha}{\mu}\right),$$

since $q_e(e) > 0$, $\frac{\partial \mathcal{E}}{\partial \alpha} \le 0$ and $V_{zz}(z) \le 0$, we have

$$\mathcal{J}_a^2(z, \alpha, A, \mu) \le 0.$$

Thus we prove part (c). ■

We sketch the proof of proposition 1 as follows. Denote constant $d < \infty$ as $d \equiv \beta^{-1} - E_{A,\mu}\mu^{-1}$. Note $Y_z(z) \geq 0$ hence $E_{A,\mu}J^2(z,\alpha,A,\mu) \geq 0$ for all z and α, so A0 is a necessary condition. Note a short-run equilibrium is given by $z = z^*, \alpha = z^*\alpha^*$ where $z^{*\backslash}$ and α^* solve $E_{A,\mu}J^1(z^*,\alpha^*,A,\mu) = d$ and $E_{A,\mu}J^2(z^*,\alpha^*,A,\mu) = d$.

We want to show, for any $\alpha \in \left[0, \min_{\mu\in\Sigma_{A,\mu}}\right]$ there exists $z \in \mathbb{R}_+$ such that $E_{A,\mu}J^2(z,\alpha,A,\mu) = d$. Suppose not for some α. Fix that α, then by the continuity of $E_{A,\mu}J^2(z,\alpha,A,\mu)$ in z and α, it is either $E_{A,\mu}J^2(z,\alpha,A,\mu) > d$ for all z or $E_{A,\mu}J^2(z,\alpha,A,\mu) < d$ for all z. Consider the first case. This implies $E_{A,\mu}J^2(z_2,\alpha,A,\mu) > d$. Note that from lemma 3(c), $E_{A,\mu}J^2(z,\alpha,A,\mu)$ is increasing in α. Then we have $E_{A,\mu}J^2\left(z_2, \min_{\mu\in\Sigma_{A,\mu}}, A, \mu\right) \geq E_{A,\mu}J^2(z_2,\alpha,A,\mu) > d$ which is contradiction to part (b) of A1. Consider the second case that $E_{A,\mu}J^2(z,\alpha,A,\mu) < d$ for all z. This implies $E_{A,\mu}J^2(z_1,\alpha,A,\mu) < d$, and hence $E_{A,\mu}J^2(z_1,0,A,\mu) \leq E_{A,\mu}J^2(z_1,\alpha,A,\mu) < d$ which is contradiction to part (a) of A1. So we prove for any $\alpha \in \left[0, \min_{\mu\in\Sigma_{A,\mu}}\right]$ there exists $z \in \mathbb{R}_+$ such that $E_{A,\mu}J^2(z,\alpha,A,\mu) = d$. The locus (z,α) solving $E_{A,\mu}J^2(z,\alpha,A,\mu) = d$ is continuous and connected. On the other hand, since $J^1(z,\alpha,A,\mu)$ is continuous in z and α, so does $E_{A,\mu}J^1(z,\alpha,A,\mu)$.

We want to prove the image of z of A is the subset of $[z_1^*, z_2^*]$. Denote A as the set of locus (z,α) solving $E_{A,\mu}J^2(z,\alpha,A,\mu) = d$. Suppose not, then consider there is $z_3 > z_2^*$ such that there exist $\alpha_3 \in \left[0, \min_{\mu\in\Sigma_{A,\mu}}\right]$ such that $E_{A,\mu}J^2(z_3,\alpha_3,A,\mu) = d$. The case $z_3 < z_1^*$ is similar and we just ignore the proof. Since from Lemma 3 we have $E_{A,\mu}J^2_z(z,\alpha,A,\mu) \geq 0$ and $E_{A,\mu}J^2_\alpha(z,\alpha,A,\mu) \leq 0$, by continuity of $E_{A,\mu}J^2(z,\alpha,A,\mu)$, then we have $E_{A,\mu}J^2(z_3,\alpha_3,A,\mu) \geq E_{A,\mu}J^2\left(z_3, \min_{\mu\in\Sigma_{A,\mu}}, A, \mu\right) \geq E_{A,\mu}J^2\left(z_2^*, \min_{\mu\in\Sigma_{A,\mu}}, A, \mu\right) = d$. This holds only if $E_{A,\mu}J^2\left(z_3, \min_{\mu\in\Sigma_{A,\mu}}, A, \mu\right) = d$, which contradicts to the premise that z_2^* is the supremum solving $E_{A,\mu}J^2\left(z_2, \min_{\mu\in\Sigma_{A,\mu}}, A, \mu\right) = d$. So we prove the image of z of A is the subset of $[z_1^*, z_2^*]$. This implies the solution $z^* \in [z_1^*, z_2^*]$.

We want to show the existence of the solution $(z^*,\alpha^*) \in A$ solving $E_{A,\mu}J^1(z^*,\alpha^*,A,\mu) = d$. Note $(z_1, 0) \in A$ and $\left(z_2, \min_{\mu\in\Sigma_{A,\mu}}\right) \in A$. Note $z_1 < \infty$, otherwise, we have $0 = Y_z(z_1; a) = E_{A,\mu}J^2(z_1,0,A,\mu)$ which is contradiction. Since U satisfies Inada condition, we always $E_{A,\mu}J^1(z_1,0,A,\mu) = \infty > d$. Note we always have $E_{A,\mu}J^1_\alpha(z,\alpha,A,\mu) \leq 0$, then from part (c) of A1 we have $d \geq E_{A,\mu}J^1(z_2,a_1,A,\mu) \geq E_{A,\mu}J^1\left(z_2, \min_{\mu\in\Sigma_{A,\mu}}, A, \mu\right)$. In sum, there are some (z,α) in A such that $E_{A,\mu}J^1(z,\alpha,A,\mu) > d$, and some (z,α) in A such that $E_{A,\mu}J^1(z,\alpha,A,\mu) \leq d$.Given the continuity of $E_{A,\mu}J^1(z,\alpha,A,\mu)$, since A is connected, there exists $(z^*,\alpha^*) \in A$ such that $E_{A,\mu}J^1(z^*,\alpha^*,A,\mu) = d$. That is the intersection illustrated in Figure 2.

Note given A2, the case $E_{A,\mu}\mu^{-1} = 1/\beta$ hence $d = 0$ is ruled out, hence the case that $Y_z\left(z^*\left(1 - \alpha^*/\mu\right); A\right) = 0$ for generic (A, μ) is also ruled out. From part (b) and part (c) of lemma 3, we have the locus of $E_{A,\mu}J^2\left(z, \alpha, A, \mu\right) = d$ positive slope. Also note $E\left(z, \alpha, A, \mu\right)$ is decreasing in α hence $J^1_\alpha\left(z, \alpha, A, \mu\right) \leq 0$. Again since the case $d = 0$ is ruled out, hence the case that $V_z\left(z^*\alpha^*/\mu; A\right) = 0$ for generic (A, μ) is also ruled out. Hence we always have $E_{A,\mu}J^1_\alpha\left(z, \alpha, A, \mu\right) < 0$. By the first part of A2 we have $E_{A,\mu}J^1_z\left(z, \alpha, A, \mu\right) \leq 0$, hence the locus of $E_{A,\mu}J^1\left(z, \alpha, A, \mu\right) = d$ non-positive slope. Therefore there is unique intersection.

7.2 Proof of Proposition 2

Recall that the short-run fluctuation of the economy can be characterized by the first order conditions of search intensity and nominal bond demand

$$\kappa = q_e\left(e\right)Y\left(z - \frac{\alpha}{\mu}; A\right), \tag{35}$$

$$i = q\left(e\right)Y_z\left(z - \frac{\alpha}{\mu}; A\right), \tag{36}$$

At the equilibrium, inflation rate, unemployment rate u and intensive margin of labor demand h are

$$\pi = \mu, u = 1 - q\left(e\right), h = \arg\max_{h \leq z - \frac{\alpha}{\mu}}\left\{Af\left(h\right) - h\right\}.$$

So the intensive margin of labor demand solves

$$h = \min\left\{z - \frac{\alpha}{\mu}, f_h^{-1}\left(A^{-1}\right)\right\},$$

which is increasing in μ and A.

Total differentiate (35) with respect to μ, we have

$$\frac{\partial e}{\partial \mu} = -\frac{\alpha}{\mu^2}\frac{q_e\left(e\right)^2}{\kappa q_{ee}\left(e\right)q\left(e\right)}i \geq 0,$$

which is positive since $q_{ee}\left(e\right) < 0 < q_e\left(e\right)$. So we have $dq\left(e\right)/d\mu = q_e\left(e\right)de/d\mu > 0$. Total differentiate (35) with respect to A, we have

$$\frac{\partial e}{\partial A} = -\frac{q_e\left(e\right)Y_A\left(z - \frac{\alpha}{\mu}\right)}{q_{ee}\left(e\right)Y\left(z - \frac{\alpha}{\mu}\right)} \geq 0.$$

So the extensive margin of labor demand $q\left(e\right)$ is increasing in μ and A, hence the results on unemployment rate u follow.

Total differentiate (36) with respect to A, we have

$$\frac{\partial i}{\partial A} = q_e(e) Y_z\left(z - \frac{\alpha}{\mu}\right)\frac{\partial e}{\partial A} + q(e) Y_{zA}\left(z - \frac{\alpha}{\mu}\right) \geq 0,$$

which proves equilibrium nominal interest rate i is increasing in productivity A.

Let denote the Jacobian $J(e, z)$ of the system of equations (35) and (36) as:

$$J(e, z) \equiv \begin{pmatrix} q_{ee}(e) Y(z) & q_e(e) Y_z(z) \\ q_e(e) Y_z(z) & q(e) Y_{zz}(z) \end{pmatrix}.$$

The second order necessary condition in the equilibrium implies

$$\det\left(J\left(e, z - \frac{\alpha}{\mu}\right)\right) \geq 0,$$

Total differentiate equilibrium first order conditions (35) and (36) with respect to i, we have

$$\begin{pmatrix} 0 \\ 1 \end{pmatrix} \partial i = J\left(e, z - \frac{\alpha}{\mu}\right)\begin{pmatrix} \partial e \\ \frac{\alpha}{\mu^2}\partial\mu \end{pmatrix},$$

$$\Rightarrow \frac{\partial i}{\partial \mu} = \frac{\alpha}{\mu^2}\frac{\det\left(J\left(e, z - \frac{\alpha}{\mu}\right)\right)}{q_{ee}(e) Y\left(z - \frac{\alpha}{\mu}\right)} \leq 0,$$

which proves the liquidity effect where equilibrium nominal interest rate i is decreasing in short-run money growth factor μ.

7.3 Proof of Lemma 1

Recall:

$$\mathcal{J}^2(z, \alpha, A, \mu) \equiv \frac{\beta}{\mu}\left[i\left(z\left(1 - \frac{\alpha}{\mu}\right), A\right) + 1\right]. \tag{37}$$

where $i(z, A)$ is given by

$$i = q(\mathcal{E}(z, A)) Y_z(z; A)$$

$E(z, A)$ solves

$$\kappa = q_e(\mathcal{E}) Y(z; A). \tag{38}$$

So we have

$$\mathcal{J}^2_\mu(z, \alpha, A, \mu) = -\frac{\beta}{\mu^2}\left[i\left(z\left(1 - \frac{\alpha}{\mu}\right), A\right) + 1\right] + \frac{\beta z \alpha}{\mu^3} i_z\left(z\left(1 - \frac{\alpha}{\mu}\right), A\right) < 0,$$

since by lemma 3 we have $i_z < 0$.

7.4 Proof of Lemma 2

Note liquidity effect happens when $i = 0$, where i is determined by (16) and (17). So $i = 0$ happens when either $q(e) = 0$ or $Y_z\left(z - \frac{\alpha}{\mu}; A\right) = 0$. $q(e) = 0$ implies $q_e(e) = \infty$. Since $Y\left(z - \frac{\alpha}{\mu}; A\right)$ is finite so $q(e) = 0$ is ruled out by (16). So liquidity trap happens when $Y_z\left(z - \frac{\alpha}{\mu}; A\right) = 0$. Express $Y\left(z - \frac{\alpha}{\mu}\right)$ as

$$Y\left(z - \frac{\alpha}{\mu}\right) = Af\left(h\left(z - \frac{\alpha}{\mu}\right)\right) - h\left(z - \frac{\alpha}{\mu}\right),$$

where $h(z)$ is given by

$$h(z) = \min\left\{z, f_h^{-1}\left(A^{-1}\right)\right\}.$$

So $Y_z\left(z - \frac{\alpha}{\mu}; A\right) = 0$ iff $h_z\left(z - \frac{\alpha}{\mu}; A\right) = 0$, which is the case when either productivity A is sufficiently low, or money growth factor μ is sufficiently high.

7.5 Proof of Proposition 3:

Recall the second order necessary condition of search intensity and nominal bond demand in the equilibrium implies

$$\det\left(J\left(e, z - \frac{\alpha}{\mu}\right)\right) > 0.$$

The long-run economy is characterized by

$$\kappa = q_e(e)\, Y\left(z - \frac{\alpha}{\mu}; A\right), \tag{39}$$

$$\frac{\mu}{\beta} - 1 = q(e)\, Y_z\left(z - \frac{\alpha}{\mu}; A\right), \tag{40}$$

Total differentiate the above system, we have

$$J\left(e, z - \frac{\alpha}{\mu}\right)\begin{pmatrix} \partial e \\ \partial\left(z - \frac{\alpha}{\mu}\right)\end{pmatrix} = \begin{pmatrix} 0 \\ \beta^{-1}\end{pmatrix}\partial\mu - \begin{pmatrix} q_e(e)\, Y_A\left(z - \frac{\alpha}{\mu}\right) \\ q(e)\, Y_{zA}\left(z - \frac{\alpha}{\mu}\right)\end{pmatrix} dA.$$

So by Crammer rule

$$\frac{\partial e}{\partial \mu} = -\frac{q_e(e)\, Y_z\left(z - \frac{\alpha}{\mu}\right)}{\beta \det\left(J\left(e, z - \frac{\alpha}{\mu}\right)\right)} \leq 0, \quad \frac{\partial}{\partial \mu}\left(z - \frac{\alpha}{\mu}\right) = \frac{\pi^2}{\beta\alpha}\frac{q_{ee}(e)\, Y\left(z - \frac{\alpha}{\mu}\right)}{\det\left(J\left(e, z - \frac{\alpha}{\mu}\right)\right)} \leq 0,$$

$$\frac{\partial e}{\partial A} = q_e(e)\, q(e)\frac{Y_{zA}\left(z - \frac{\alpha}{\pi}\right) Y_z\left(z - \frac{\alpha}{\pi}\right) - Y_A\left(z - \frac{\alpha}{\pi}\right) Y_{zz}\left(z - \frac{\alpha}{\pi}\right)}{\det\left(J\left(e, z - \frac{\alpha}{\pi}\right)\right)} \geq 0,$$

$$\frac{\partial}{\partial A}\left(z - \frac{\alpha}{\mu}\right) = \frac{q_e(e)^2\, Y_z\left(z - \frac{\alpha}{\pi}\right) Y_A\left(z - \frac{\alpha}{\pi}\right) - q_{ee}(e)\, q(e)\, Y\left(z - \frac{\alpha}{\pi}\right) Y_{zA}\left(z - \frac{\alpha}{\pi}\right)}{\det\left(J\left(e, z - \frac{\alpha}{\pi}\right)\right)} \geq 0.$$

On other hand, combining the first order conditions of consumption money and of production money implies:

$$V_z\left(\frac{\alpha}{\mu}\right) = Y_z\left(z - \frac{\alpha}{\mu}\right), \tag{41}$$

where $V(z)$ is given by

$$V(z) \equiv \max_{c \leq z}\{U(c) - c\},$$

so it is straight-forward to show

$$V_z(z) \geq 0, V_{zz}(z) \leq 0.$$

So we have

$$\frac{\partial(\alpha/\mu)}{\partial\mu} = \frac{Y_{zz}\left(z - \frac{\alpha}{\mu}\right)}{V_{zz}\left(\frac{\alpha}{\mu}\right)}\frac{\partial}{\partial\mu}\left(z - \frac{\alpha}{\mu}\right) \leq 0, \frac{\partial(\alpha/\mu)}{\partial A} = \frac{Y_{zz}\left(z - \frac{\alpha}{\mu}\right)}{V_{zz}\left(\frac{\alpha}{\mu}\right)}\frac{\partial}{\partial A}\left(z - \frac{\alpha}{\mu}\right) \geq 0.$$

And hence the comparative statics on economy liquidity z is

$$\frac{\partial z}{\partial\mu} = \frac{\partial}{\partial\mu}\left(z - \frac{\alpha}{\mu}\right) + \frac{\partial(\alpha/\mu)}{\partial\mu} \leq 0, \frac{\partial z}{\partial A} = \frac{\partial}{\partial A}\left(z - \frac{\alpha}{\mu}\right) + \frac{\partial(\alpha/\mu)}{\partial A} \geq 0.$$

7.6 Proof of Proposition 4

Assume equilibrium and optimal money growth factor, denoted as $\{\mu_s\}_{s=t}^{\infty}$, exist. Denote the value of household under $\{\mu_s\}_{s=t}^{\infty}$ as W_t^{CM}, where dependence on N_t is suppressed. Given N_t, define W^* as

$$W^*(\varepsilon) \equiv \max_{z_{t-1}^P, z_{t-1}^C}\left\{-z_{t-1}^P - z_{t-1}^C + \beta\mathbb{E}_{t-1}\max_{e_t \geq 0}\left\{\begin{array}{c}(1-N_t)\,Y\left(\frac{z_{t-1}^P}{\pi_t} + \Delta_t + \varepsilon_t; A_t\right) \\ +N_t\left[q(e_t)\,Y\left(\frac{z_{t-1}^P}{\pi_t} + \Delta_t + \varepsilon_t; A_t\right) - \kappa e_t\right] \\ +\frac{z_{t-1}^P}{\pi_t} + n_t V\left(\frac{z_{t-1}^C}{\pi_t}\right) + \frac{z_{t-1}^C}{\pi_t} + W_t^{CM}(N_{t+1})\end{array}\right\}\right\},$$

where Δ_t is the supporting money injection under $\{\mu_s\}_{s=t}^{\infty}$. We have $\sup_\varepsilon W^*(\varepsilon) \geq W_t^{CM}$, since $W^*(\mathbf{0}) = W_t^{CM}$. The first order condition of ε_t is

$$Y_z\left(\frac{z_{t-1}^P}{\pi_t} + \Delta_t + \varepsilon_t; A_t\right) = 0. \tag{42}$$

Hence the first order condition of z_{t-1}^P is

$$1 = \beta\mathbb{E}_{t-1}\left[\frac{1}{\pi_t}\left\{[1 - N_t + q(e_t)\,N_t]\,Y_z\left(\frac{z_{t-1}^P}{\pi_t} + \Delta_t + \varepsilon_t; A_t\right) + 1\right\}\right] = \beta\mathbb{E}_{t-1}\left(\frac{1}{\pi_t}\right), \tag{43}$$

Hence the first order condition of z_{t-1}^C must imply

$$V_z\left(\frac{z_{t-1}^C}{\pi_t}\right) = 0. \tag{44}$$

Obviously, z_t^* and α^* stated in proposition 4 constitute an equilibrium under $\{\mu_s^*\}_{s=t}^{\infty}$ with $z_{t-1}^C = \beta\alpha^*$ and $\pi_t = \beta$, as well as satisfy (42) and (44). Then one can show z_t^* and α^* attain $\sup_\varepsilon W^*(\varepsilon)$ under $\{\mu_s^*\}_{s=t}^{\infty}$, due to the concavity of $W^*(\varepsilon)$. So $\{\mu_s^*\}_{s=t}^{\infty}$ must maximize W_t^{CM}.

8 Appendix B: Details of Data

Below we list the details of each of the monthly series used in the calibration exercise, with the FRED Series IDs. All data are seasonally adjusted.

- Output: The real gross domestic product (GDPC96). The monthly real GDP is estimated from the linear spline of the quarterly series.
- Productivity: The ratio of the real gross domestic product (GDPC96) over the product of all employees,total private industries (USPRIV) and the average weekly hours of production and nonsupervisory employees total private (AWHNONAG).
- Unemployment: The share of the unemployed in the labor force corresponses to the civilian unemployment rate (UNRATE). For exmaple 5% unemployment rate in the data is taken as 0.05
- Inflation factor: The gross monthly growth factor of the consumer price index for all urban consumers all items (CPIAUCSL).
- Money growth factor: The gross monthly growth factor of the sweep-adjusted M1, from Cynamon et al (2006)
- Nominal Interest factor: The gross factor corresponses to the Moody's seasoned AAA corporate bond yield (AAA). For exmaple 12% yield rate in the data is taken as 1.12
- Saving share: The gross factor corresponse to the personal saving rate (PSAVERT). For exmaple 7% saving rate in the data is taken as 0.07

9 Appendix C: Steady State for Calibration

For use of calibration, consider a deterministic steady state with no shocks, ie, $A_t = A$ and $\mu_t = \mu$. Then (30) implies $\pi = \mu$, (24) implies

$$u = \frac{\sigma}{q(e) + \sigma},$$

and the first order condition of bond demand (31) in the steady state implies $i = \mu/\beta$.The expected surplus difference between a filled firm and a vacant firm in the steady state becomes

$$\Gamma \equiv \frac{\kappa e + [1 - \sigma - q(e)] Y \left(z - \frac{z^C}{\mu}; A\right)}{1 - \beta [1 - \sigma - q(e)]}.$$

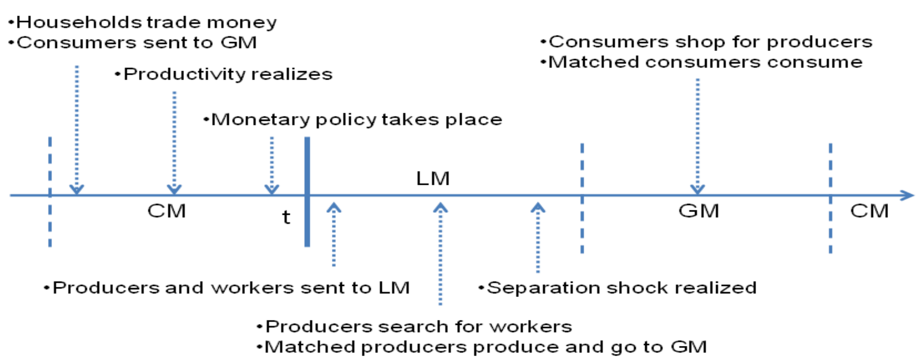
The equilibrium first order condition of search intensity (27) in the steady state implies

$$\kappa = q_e(e)\left[\frac{\beta\kappa e + Y\left(z - \frac{z^C}{\mu}; A\right)}{1-\beta\left[1-\sigma-q(e)\right]}\right].$$

The first order conditions of money demand (25) and (26) in the steady state implies

$$\frac{\mu}{\beta} - 1 = \frac{q(e)}{q(e)+\sigma} Y_z\left(z - \frac{z^C}{\mu}; A\right),$$

$$V_z\left(\frac{z^C}{\mu}\right) = Y_z\left(z - \frac{z^C}{\mu}; A\right).$$



Figre 1 Timeline of events within a period

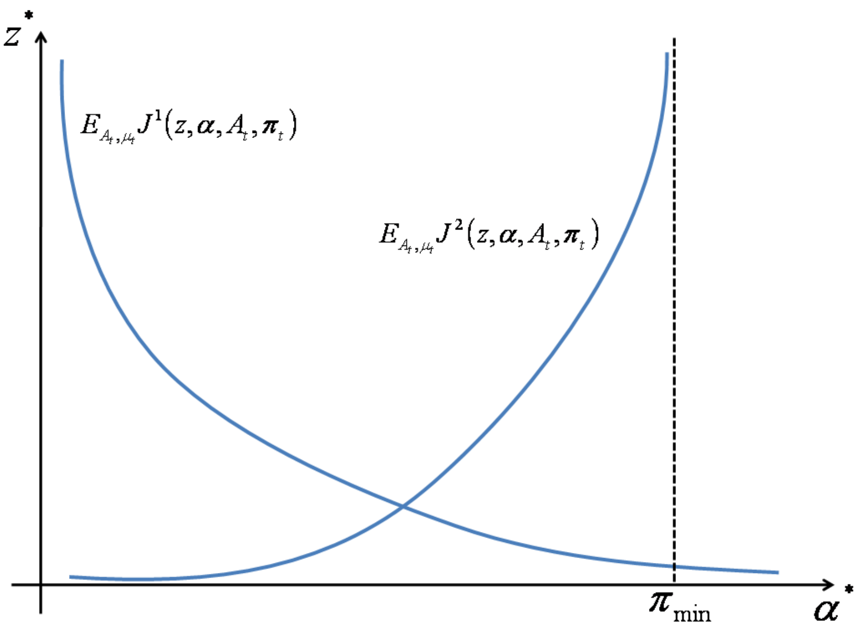


Figure 2 Loci of $\mathbb{E}\mathcal{J}^1(z^*, z^*\alpha^*, A, \mu) = \beta^{-1} - \mathbb{E}\mu^{-1}$ and
$\mathbb{E}\mathcal{J}^2(z^*, z^*\alpha^*, A, \mu) = \beta^{-1} - \mathbb{E}\mu^{-1}$

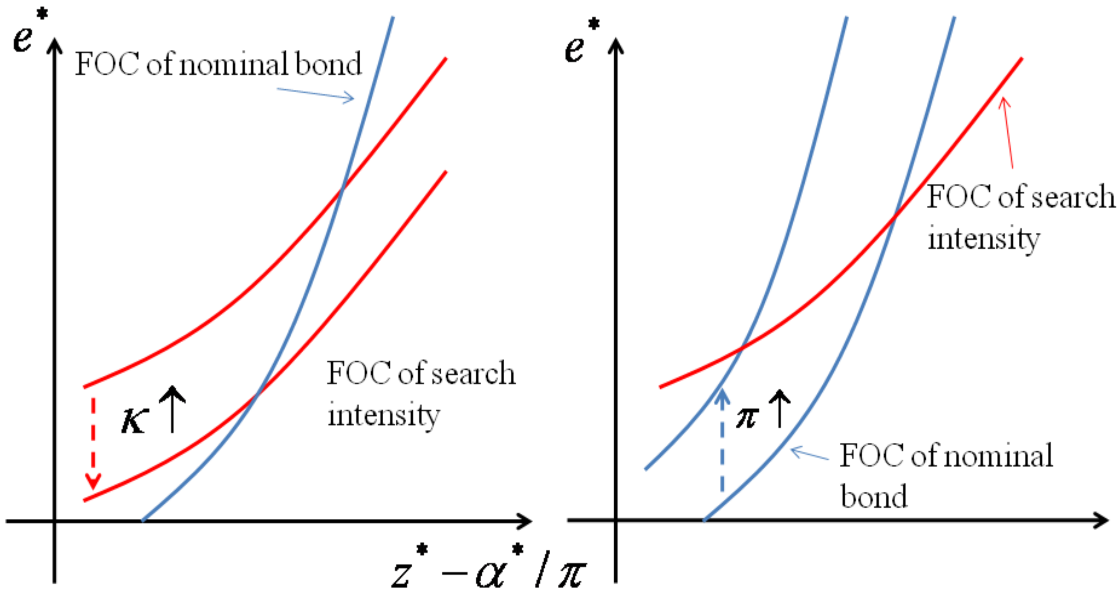


Figure 3 Plot of first order conditions of nominal bond and of search intensity.

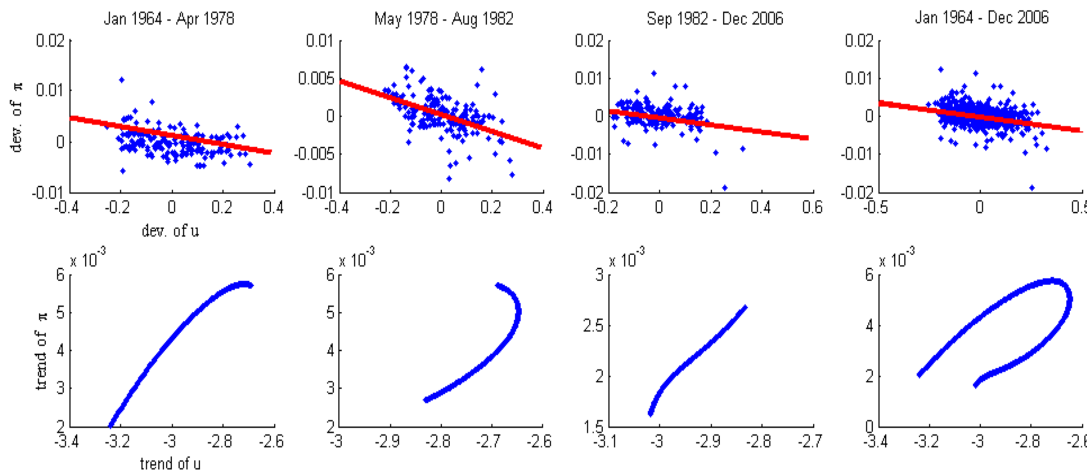


Figure 4 Plot of decompositions of inflation rate and of unemployment rate during different period.

The time series of the monthly CPI inflation rate and unemployment rate span from 1964:Jan to 2006:Dec, such that effects of the recent crisis is robustly avioded. The sample is equally divided into three subsamples: 1964:Jan to 1978:Apr, 1978:May to 1992:Aug and 1992:Oct to 2006:Dec. The upper panel plots log-deviation from the HP filtered trends, with smoothing parameter 16×30^4 , in the three subsamples and the entire sample. The lower panel plots the HP filtered trends, with smoothing parameter 16×30^4. Source: Federal Reserve Bank of St Louis.

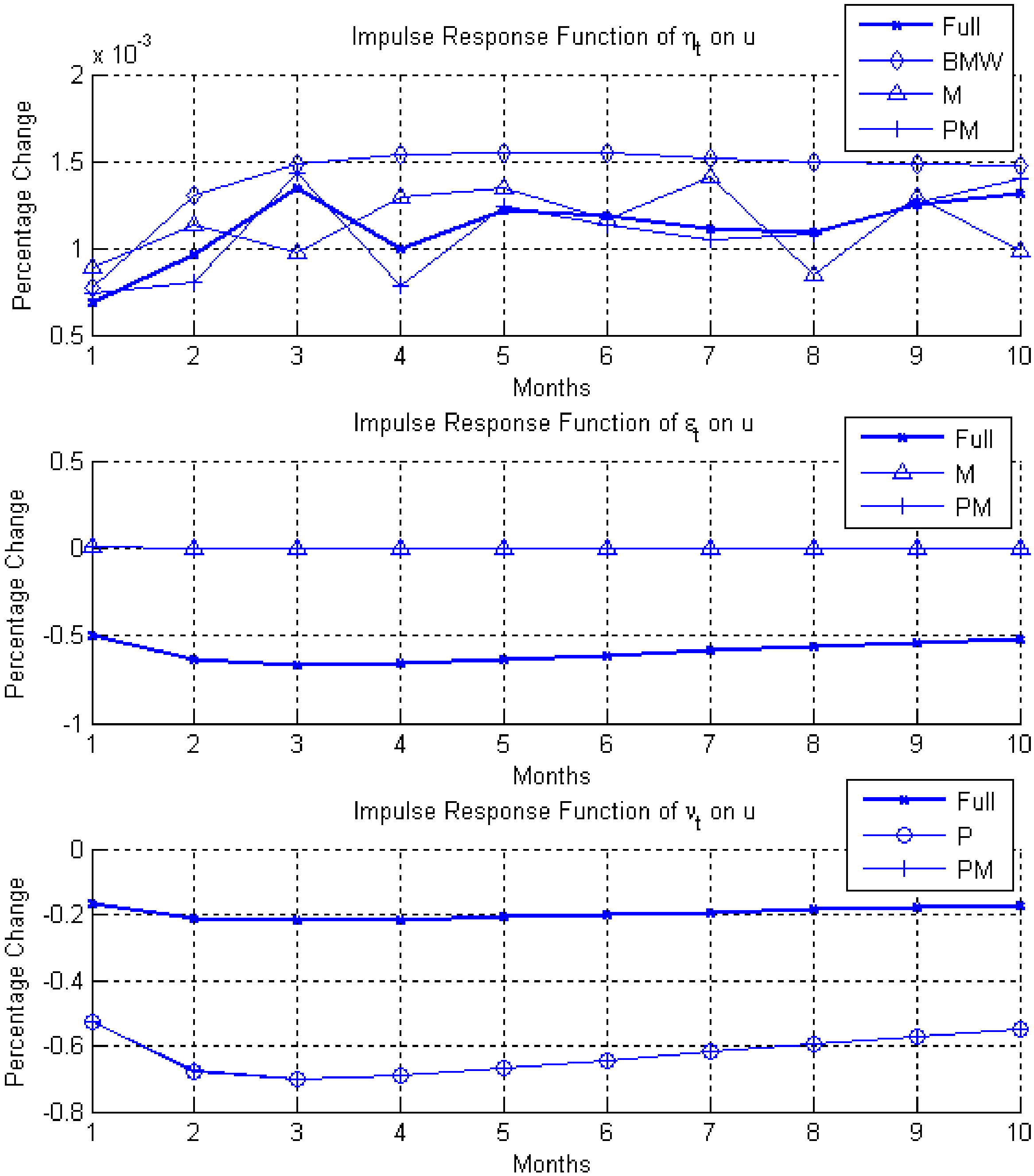


Fig 5 Impulse response functions on the unemployment rate.

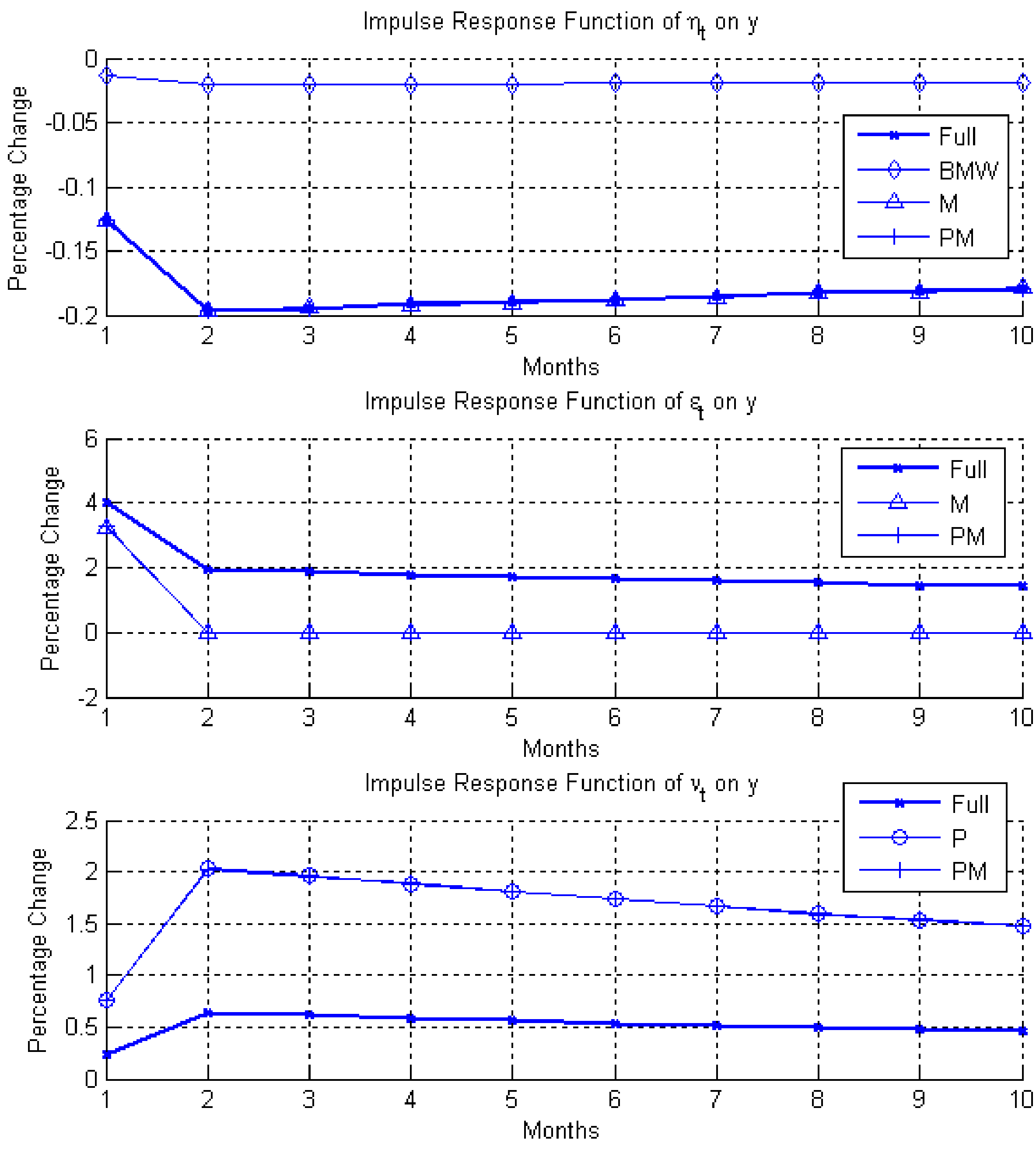


Fig 6 Impulse response functions on output.

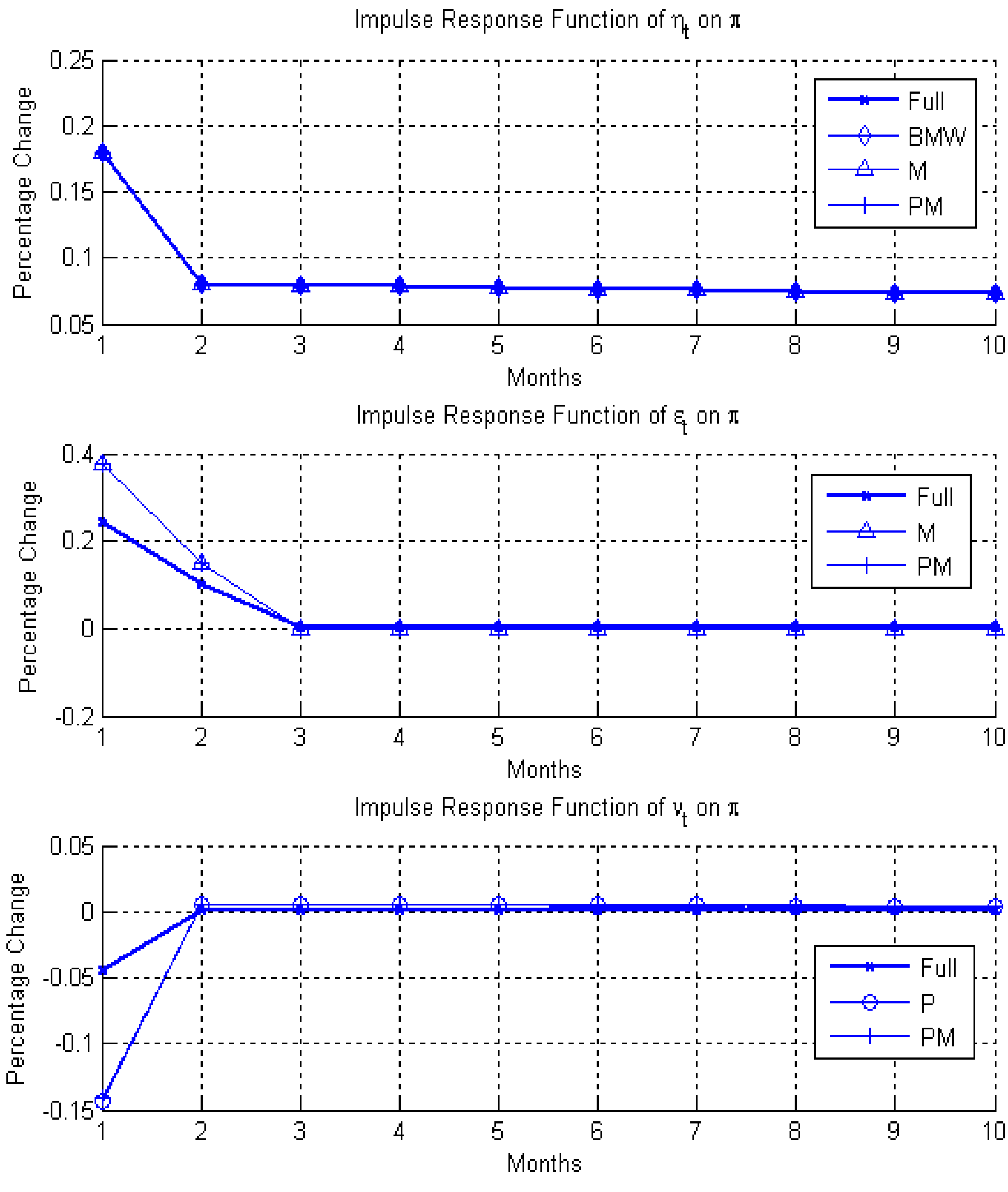


Fig 7 Impulse response functions on the inflation rate.

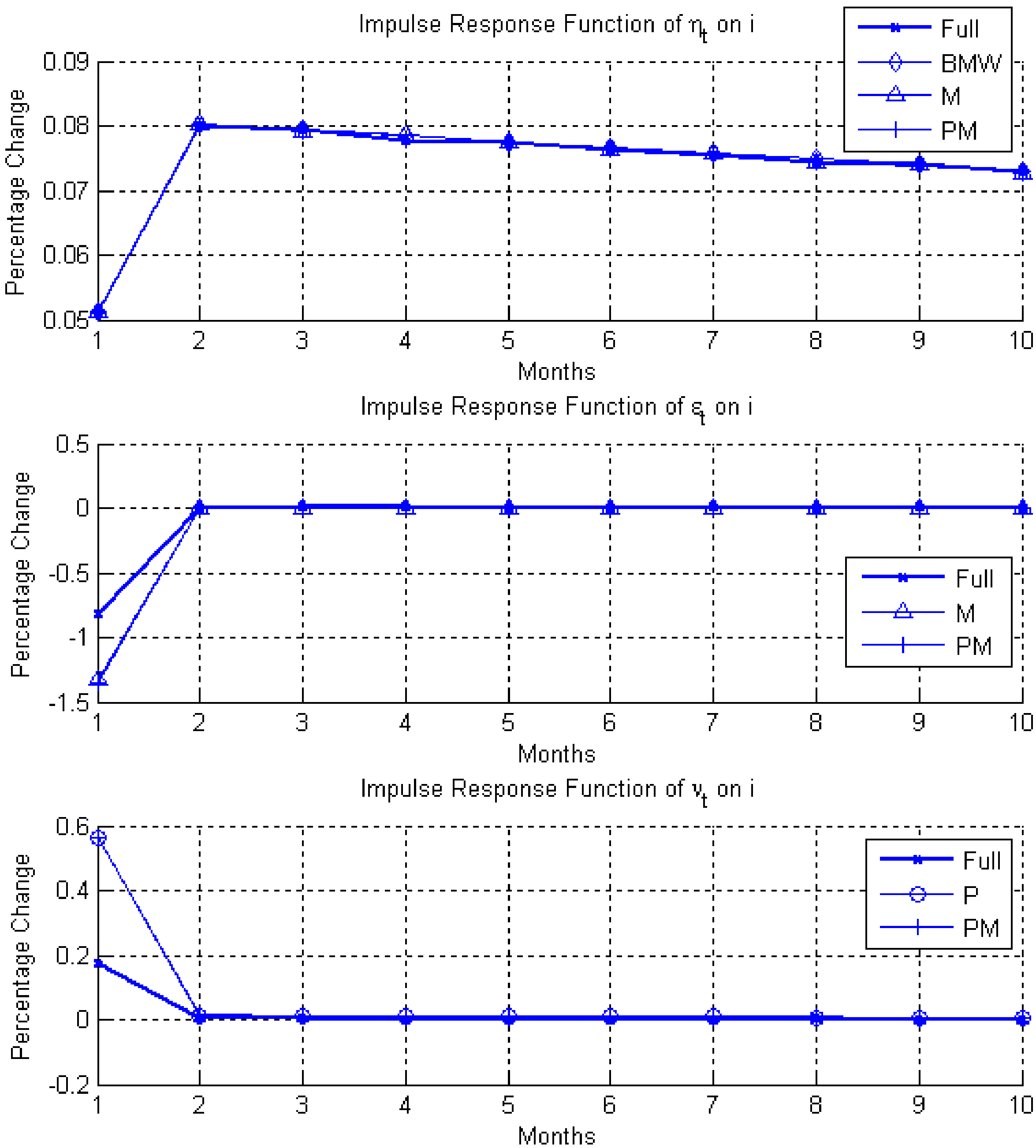


Fig 8 Impulse response functions on the nominal interest rate.